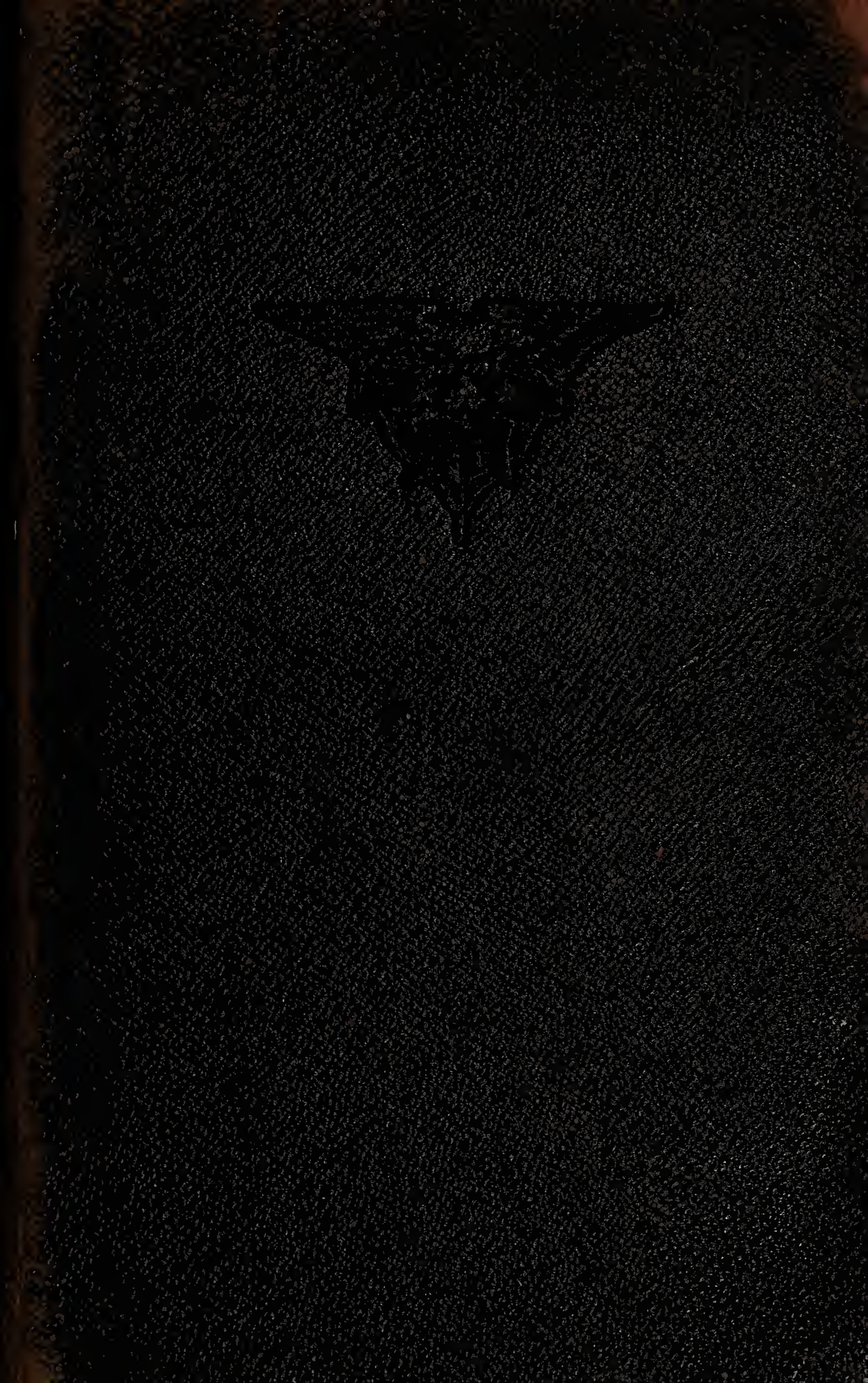

LIBRARY

BUREAU OF EDUCATION

LT
QA37
B25

6—1132

NLE
ed.gov

Modern American Education

A SERIES OF TEXTS PREPARED FOR THE AMERICAN EDUCATIONAL INSTITUTE AS PART OF ITS MODERN AMERICAN EDUCATION COURSE

LIBRARY
FEB 26 1921

AMERICAN EDUCATIONAL INSTITUTE, Inc.
PHILADELPHIA

Modern American Education

A SERIES OF TWELVE TEXTS PREPARED FOR THE AMERICAN EDUCATIONAL INSTITUTE

EDITOR-IN-CHIEF

PETER P. WAHLSTAD

Titles:	Department Editors:
ENGLISH LANGUAGE AND GRAMMAR	PETER P. WAHLSTAD
MATHEMATICS AND PHYSICS	MAURICE J. BABB CHARLES B. BAZZONI
GEOGRAPHY AND GEOLOGY	ROBERT M. BROWN GEO. McC. PRICE
AMERICAN AND MODERN EUROPEAN HISTORY	WILBER L. STONEX
PHYSIOLOGY AND HYGIENE	GEO. VAN N. DEARBORN
NATURE STUDY AND ASTRONOMY	GEO. McC. PRICE ERIC DOOLITTLE
SOCIAL SCIENCE	HENRY P. FAIRFIELD
HOME ECONOMICS AND CHEMISTRY	EDITH H. LAVELL DAVID W. HORN
PSYCHOLOGY AND CHILD STUDY	ARTHUR J. JONES
ENGLISH AND AMERICAN LITERATURE	ROBERT M. LOVETT
AMERICAN GOVERNMENT	P. ORMAN RAY
COMMERCE AND INDUSTRY	THURMAN W. VAN METRE

AMERICAN EDUCATIONAL INSTITUTE, Inc.

PHILADELPHIA

Mathematics and Physics

PART I: MATHEMATICS

EDITED BY
MAURICE J. BABB, Ph. D.
Assistant Professor of Mathematics in the
University of Pennsylvania

PART II: PHYSICS

EDITED BY
CHARLES B. BAZZONI, Ph. D
Assistant Professor of Physics in the
University of Pennsylvania

MODERN AMERICAN EDUCATION
VOLUME II

AMERICAN EDUCATIONAL INSTITUTE, Inc.
PHILADELPHIA

LT
QA37
.B25

COPYRIGHT 1920 BY
AMERICAN EDUCATIONAL INSTITUTE, INC.

COPYRIGHT IN GREAT BRITAIN 1920 BY
AMERICAN EDUCATIONAL INSTITUTE

ALL RIGHTS RESERVED

JAN 20 1921

©CLA608094

778782

EDITOR'S PREFACE TO MATHEMATICS

Tho mathematics as ordinarily taught in the lower grades of our public schools is well enough presented, many have found that its later presentation is open to just criticism on the ground that the treatment is too abstract and that its several parts are kept in "water-tight compartments," as tho geometry and algebra and calculus were entirely separated from arithmetic. Attempts which have been made to combine them have, however, not been uniformly successful.

In the present Text an attempt has been made to develop the main thoughts in elementary mathematics in such a way as to show the relation of its various subjects to what we call *number* as expressed in a power series, and thereby trace their relation to each other.

The main purpose of mathematics is to formulate a concise and suggestive form of notation in which various problems may be so stated that we readily comprehend the relations expressed. By way of differentiation it is well to bear in mind that arithmetic concerns itself mainly with countable things, while geometry deals with uncountable things, taking note, however, of relative shape, size, position and motion. Again trigonometry may be said to *compute* where geometry *constructs*.

Algebra generalizes number and makes it possible to present analysis in connection with geometry in algebraic geometry. Finally calculus gives us the means of expressing the relations of variable quantities at any certain instant.

The scope of the present work does not permit us to pursue our study of the various divisions of the subject to include all that mathematical science has collected and classified under the various heads. Special reference works on the subject may usually be consulted in all good libraries. Among such works may be mentioned: Hilbert's *Foundation of Geometry;* Chrystal's *Algebra;* Hobson's *Trigonometry;* Salmon's *Analytic Geometry,* and Price's or Williamson's *Calculus.*

But for the purposes of the ordinary subscriber the present Text will probably be found sufficiently comprehensive. A warning should be sounded against attempting to read it too hurriedly. Mathematics appeals to the reasoning powers—it calls for thinking and cannot be understood by the superficial reader. It is not required, nor intended, that this Text should be covered thoroly in the time allotted to its reading in the general reading assignment, tho a very comprehensive grasp of its main portions may be obtained within that time, especially if the reader makes the extra effort justified in this case. The subject, in order that it may be fully assimilated, should be carried along with the reading of the next Text (Physics) and indeed taken up again from time to time for review and further study. When so read, no great difficulty should be experienced by the diligent and deliberate students of this Text.

MAURICE J. BABB.

August, 1920.

TABLE OF CONTENTS

CHAPTER I

FUNDAMENTAL PRINCIPLES OF ARITHMETIC

SECTION		PAGE
1.	Primitive Notation	1
2.	Representation of Numbers	2
3.	The Decimal Point	4
4.	Addition	4
5.	Subtraction	6
6.	Multiplication	7
7.	The Decimal Point in Multiplication	8
8.	Division	9
9.	Decimal Point in Division	11
10.	Correct Measurement	11
11.	Care in Computing Measurements	12
12.	Divisibility	14
13.	Prime Numbers	15
14.	Rules of Exact Division	15
15.	Greatest Common Factors	16
16.	Multiples	17
17.	Cancellation	18
18.	Casting Out Nines	19

CHAPTER II

FRACTIONS

SECTION		PAGE
1.	Proper and Improper Fractions	21
2.	Mixed Numbers	22
3.	Common Denominator	23
4.	Multiplication of Fractions	24
5.	Division of Fractions	25
6.	Denominate Numbers	27

7. Tables of Denominate Numbers 27
8. Facts About the Tables 29
9. Exercises in Denominate Numbers 30
10. Operations in Compound Numbers 32
11. The Metric System 32

CHAPTER III

PERCENTAGE AND APPLICATION

SECTION PAGE
1. Percentage 35
2. Commercial Discount 37
3. Profit and Loss 38
4. Commission 39
5. Simple Interest 39
6. General Method of Computing Interest 42
7. Exact Interest 43
8. Compound Interest 43

CHAPTER IV

PLANE GEOMETRY—LINES AND ANGLES

SECTION PAGE
1. Properties of Space 47
2. Dimensions 48
3. Geometric Elements 49
4. Formal Geometry 50
5. The Geometrician's Tools 51
6. Elementary Figures—Angles 52
7. Elementary Figures—the Circle 53
8. Rigid and Non-Rigid Figures 54
9. Sum of the Angles of Triangles 55
10. Sum of the Angles of Other Figures 56
11. The Parallel Rulers 58
12. Bisectors and Perpendiculars 59
13. Drawing Tangent from External Points 60
14. Points at Which Two Circles Meet 61

15. Perpendicular from Outside a Line..... 62
16. Bisecting Lines with Parallel Lines..... 62
17. Dividing Line with Rectangular Scale..... 63
18. Concurrent Lines in Triangles..... 64
19. Symmetry 65

CHAPTER V

PLANE GEOMETRY—AREAS AND LOCI

SECTION PAGE
1. Principle of Area..... 66
2. Area of Parallelogram..... 66
3. Area of Triangles..... 67
4. Area of the Trapezoid..... 67
5. Area of Trapezium..... 68
6. Equivalent Parallelograms..... 69
7. The Square of the Hypotenuse..... 70
8. Pythagoras' Proof..... 70
9. Square Equivalent to Rectangle..... 71
10. Similar Triangles..... 72
11. Demonstration of Principles of Similarity..... 73
12. Range Finders..... 74
13. Enlarging Figures..... 75
14. Comparison of Areas of Similar Triangles..... 75
15. Ratio 76
16. Proportion 77
17. Various Forms of Proportions..... 78
18. Dividing Triangles..... 79
19. Exterior Division of a Line..... 80
20. Constructions 81
21. Dividing a Line Into Mean Sections..... 82
22. Similar Polygons..... 83
23. Regular Polygons..... 84
24. Area of a Circle..... 85
25. Loci of Points and Sets of Lines..... 87
26. Intersection of Loci..... 89
27. The Circle as Locus..... 90

CHAPTER VI

SOLID GEOMETRY

SECTION		PAGE
1.	Lines and Planes	92
2.	Dihedral Angles	93
3.	Parallel Planes	94
4.	Polyhedral Angles	96
5.	Prisms, Parallelopipeds and Other Polyhedrons	98
6.	Pyramids	100
7.	The Prismoidal Formula	104
8.	Similar Solids	104
9.	Spherical Geometry	107
10.	Figures on the Surface of a Sphere	108
11.	Angles of Spherical Triangles	109
12.	Areas of Spherical Triangles	111

CHAPTER VII

ALGEBRA—NUMBER SYSTEM

SECTION		PAGE
1.	New Kinds of Number	114
2.	The Quality of Positive and Negative	115
3.	The Principle of Permanence	116
4.	Fractions a New Kind of Number	117
5.	Power, Root and Logarithm	118
6.	Irrationals and Imaginary Numbers	118
7.	Transcendental Numbers	120
8.	Constant and Variable Numbers	120
9.	Functions and Limit of a Variable	121

CHAPTER VIII

SIMPLE EQUATIONS AND FUNDAMENTAL OPERATIONS

SECTION		PAGE
1.	The Equation	123
2.	Limitations of Variables in an Equation	124

3. The Use of the Equation 125
4. Examples in Equations 126
5. A Different Type of Problem 127
6. The Coefficient 128
7. Resemblances Between Arithmetic and Algebra 128
8. Addition and Subtraction 129
9. Multiplication 130
10. Division 130
11. Place and Order in Division 131

CHAPTER IX

PRODUCTS AND ROOTS

SECTION PAGE
1. Exponents 134
2. Special Products 135
3. The Law of Binomial Expansion 137
4. Factors 138
5. Factorization of Difference of Squares and of Cubes . . 139
6. The Factor Theorem 141
7. Highest Common Factor 143
8. Least Common Multiple 143
9. Square Root 144
10. Algebraic Example of Extracting Root 145
11. The Square Root of Numbers 146
12. Cube Root and Higher Roots 147
13. Radicals 147
14. Multiplication and Division of Radical Expressions . . 149

CHAPTER X

QUADRATICS AND SERIES

SECTION PAGE
1. Quadratics in One Unknown Quantity 152
2. Completing the Square 153
3. The General Quadratic 154
4. Relations Between the Roots 156
5. Nature of Roots of Quadratics 157

6. Higher Equations Worked Out Like Quadratics..... 158
7. Approximation 160
8. Irrational Equations..... 161
9. Exercises in Equations..... 162
10. Series 163
11. Arithmetical Progression..... 164
12. Geometric Progression..... 165
13. Circulating Decimals..... 166
14. Geometric Series in Compound Interest..... 167
15. Permutations 167
16. Combinations 168
17. Binomial Theorem..... 170
18. Convergency 172
19. Computing Value of Series..... 174

CHAPTER XI

LOGARITHMS

SECTION PAGE
1. The Invention of Logarithms..... 177
2. Logarithms to the Base 10..... 178
3. Antilogarithms 180
4. Computations 181
5. Tables of Trigonometric Functions..... 182
6. Co-Logarithms 183

CHAPTER XII

TRIGONOMETRY

SECTION PAGE
1. Directed Length..... 191
2. Measurement of Angle..... 191
3. Quadrants 193
4. Trigonometric Functions..... 194
5. Algebraic Signs of Functions..... 195
6. Seven Fundamental Formulas..... 195
7. Numerical Values of the Functions..... 198
8. Angles Greater Than 90°..... 199
9. Inverse Functions..... 201

10. Value of $\frac{\sin\theta}{\theta}$ 201
11. The Right Triangle 202
12. Solution of the Right Triangle 203
13. Solution by Logarithms 204
14. Exercises on Right Triangles 205
15. Oblique Triangles 206
16. Functions of Sum of Two Angles 207
17. Functions of Double Angle and Half Angle 209
18. Sums and Differences of Functions 210
19. The Law of Sines 211
20. Numerical Solution of Oblique Triangle by Law of Sines 211
21. The Law of Tangents 212
22. Numerical Solution by Law of Tangents 212
23. Generalized Pythagorean Theorem 214
24. Three-Side Formula 214
25. Numerical Solution by Three-Side Formula 215

CHAPTER XIII

ALGEBRAIC GEOMETRY

SECTION PAGE
1. Graphical Representation 217
2. Plotting a Graph 218
3. Principle and Use of Graphs 219
4. Systems of Equations 220
5. Solving by Algebra 221
6. Parabola $y = x^2$ 223
7. Plots of Radical Quantities 223
8. Graphs of Quadratics 225
9. The Ellipse 226
10. The Hyperbola 227
11. Simultaneous Equations 228
12. Examples in Simultaneous Equations 230
13. Special Devices 231
14. Geometry With Algebra 232
15. Synthetic Division 233
16. Theory of Equations 234
17. Relations of Roots and Coefficients 237

CHAPTER XIV

DIFFERENTIAL CALCULUS

SECTION		PAGE
1.	Fundamental Principles and Formulas	238
2.	Ratio of Increments	238
3.	Fundamental Formulas	240
4.	Examples in Fundamental Formulas	242
5.	Derivation of Formulas	243
6.	Other Forms of Notation	244
7.	Successive Differentiation	245
8.	Maxima and Minima	246
9.	Method of Working Problems	247
10.	Taylor's Theorem	248
11.	The Value of π	251
12.	Computation of Logarithms	252
13.	Logarithms of Other Numbers	254

CHAPTER XV

INTEGRAL CALCULUS

SECTION		PAGE
1.	Anti-Differentiation	255
2.	The Formula for Falling Bodies	255
3.	An Example in Interest	258
4.	Area Under a Curve	258
5.	Area of Circle	260
6.	Area Between Curves	260
7.	Length of Curve	261
8.	Volumes of Solids of Revolution	261
9.	Areas of Surfaces of Revolution	262
10.	Formulas of Integration	262

CHAPTER I

FUNDAMENTAL PRINCIPLES OF ARITHMETIC

1. Primitive Notation.—Man did not always possess the means for calculating numbers and quantities that he has today. He did not always have a multiplication table; he could not always figure fractions; and even simple sums in addition were once more than the wisest man living could do. The cave-man did not even *think* in numbers. He did not say he had *two* horses; he would say he had a black horse and a white horse, or a large horse and a small horse, but the idea of counting one, two, three, etc., was wholly foreign to him. Even at the present day, it is said of the savage tribes in remote parts of Australia that they cannot count beyond two.

Only slowly did man learn to distinguish three and four. After that, he found it necessary to count on his ten fingers. Time came, however, when an old chief, wishing to get some information about how his enemy's fighting forces compared with his own, employed a method which, tho crude, was not unlike modern enumeration. He would send three or four men to count the opposing army. The one nearest the enemy would count up to ten on his fingers and then signal the next man, who would count the tens thus delivered to him. He in turn, as soon as he had ten tens, would signal the third man, who thus took care of the hundreds. We can imagine how upon their return to camp the "hundred" man would hold up so many fingers, then how the "ten" man would show in the same manner how many tens he had and finally how the "unit"

man used his fingers to show how many he had counted after signaling the last ten. So, if the "hundred" man held up eight fingers, the "ten" man three and the "unit" man nine, the old chief would know that he had to prepare himself to meet an army of 839 men. He would count up his own men in the same way and if he found that he had to use a fourth man, he would probably not hesitate about meeting the enemy in battle.

Even in those days they had discovered the important truth that in using numbers more attention must be paid to *place* than to *size* of figures or digits. Thus in the above number (839) we know that the figure 3 represents more than three times the value of the larger figure 9, since 3 here stands for thirty (three tens) whereas three times nine make only 27. Similarly the figure 8 represents 800, a value nearly 27 times larger than that which 3 stands for.

2. Representation of Numbers.—The ancients, when they wanted to compute, used a board upon which they sprinkled sand and then drew lines in the sand. On these lines they put pebbles or drew cross-lines to represent figures. The Romans used wax tablets for the same purpose. To this day the Chinese laundryman probably depends upon this "swan pan" in making out your laundry bill, and in many cases is more expert than a bank clerk with a modern computing machine, as was shown in a contest some time ago at Portland, Oregon.

In the 10th century the Monk Gerbert, afterwards Pope Sylvester II, introduced to Europe the "Hindu-Arabic Number System," which he obtained thru the Moors in Spain. Yet the new system made very slow

progress, the Church not regarding it favorably. In 1299 Florentine merchants were forbidden to use it, and even down to about the time of our Revolutionary War ancient methods were still doggedly persisted in. It may prove surprising to learn that as late as 1783 the British Exchequer used notched sticks to keep its tax accounts. Turks and Arabs still count as they did one thousand years ago.

The zero (0) is of relatively recent origin. Even Shakespeare would have written three hundred twenty-seven $\$\!\!\not\!\!\not\!\!\not\!\!\not\!\!\not\!\!\not\!\!\not$ and five hundred six $\not\$\,|\not\!\!\not\!\!\not\!\!\not\!\!\not\!\!\not\!\!\not$, but we write 327 and 506. No one knows who originated the zero. It merely occupies a place where there is no number. The Chinese has his wire and does not bother with the zero and our ancestors who ciphered on the line had no use for it either. But we who use no line to keep our numbers straight find the zero necessary.

It is customary to set off by commas at every third figure any number of ten thousand and more. Thus we commonly write 10,000. A number of four figures, such as 7358, can be read at a glance without the assistance of a comma, which is used merely for convenience in reading and does not affect the value of the numbers. In reading or writing a larger number, it is convenient to remember that the first comma, reading from right to left, sets off the thousands, the second comma the millions, the third comma the billions and the fourth comma the trillions. Thus the number

216,216,216,216,216.

reads: Two hundred sixteen *trillions*, two hundred sixteen *billions*, two hundred sixteen *millions*, two hundred sixteen *thousand*, two hundred sixteen.

A difference should be noted here with regard to the American and the British way of counting. With us and the French a billion is one thousand millions, but the British billion has one million millions. The British billion is therefore our trillion.

3. The Decimal Point.—When in making calculations we wish to show that a figure in a number is the *unit* figure and not tens or hundreds, we frequently put a decimal point (.) after it. If we write 327., the decimal point shows that whatever figures may later be added to the right of it are less than the unit. Thus 327.00 is the same as 327. But if we move the decimal point one place to the right (3270.0), the value of each figure becomes ten times larger. Moving the point two places to the right (32700.) is equal to multiplying by one hundred; three places to the right multiplies by one thousand, and so forth.

For the same reason, when we move the decimal point one place to the *left* (32.7) we divide by ten; two places (3.27) by one hundred, and so on.

The first number to the right of the decimal point represents tenths, the second hundredths, the third thousandths, and so on. If we write 327.25, we know that we have on the left of the point three hundred twenty-seven, and on the right two tenths and five hundredths, which is the same as twenty-five hundredths and is commonly so named.

4. Addition.—In arithmetic, addition is the uniting of two or more numbers or quantities in one, called the *sum*. The sign for addition is + (plus). With practice, it is possible to add rapidly numbers of two, three and even four figures, provided, however, that

the numbers are arranged in columns with decimal point directly under decimal point. This is because, as we have seen, the relative *place* occupied by a figure is of first importance to a correct result. *Any one who wishes to master even the simplest mathematical problem must of necessity become and remain familiar with the addition of numbers less than ten to numbers less than one hundred, and also with subtraction as performed with such numbers.* Only constant practice will enable one to say instantly, even if awakened from a sound sleep, that 7 added to 28 is 35 and that 8 from 26 is 18. Any possible combination of this kind should be solved without conscious effort.

In adding numbers having decimals, we proceed precisely as when adding whole numbers, our sole care being to see that the decimal points are not out of line. If some of the numbers to be added have two or three decimals and others only one or even none, there is less likelihood of error if we put in the figure zero (0) so that each number has an equal number of decimal places. It is as important to indicate that there is no number as to say that there is one. The zero shows that we did not carelessly drop a figure. Thus if we were to add 28.675 +13.56+19.4+21., it would be well to write down the numbers as shown herewith.

```
28.675
13.560
19.400
21.000
------
82.635
```

The following types of problems give good exercise in addition and are very practical. If adding columns (vertically) and rows (horizontally) produce results that agree, the answers are correct.

8.68	4.34	6.93	19.95
5.83	4.98	5.25	16.06
4.83	5.81	2.87	13.51
19.34	15.13	15.05	49.52

5. Subtraction.—Subtraction is the inverse of addition, and involves taking one number or quantity from another. The sign for subtraction is − (minus). In the example $6-4=2$, which reads 6 minus 4 equals 2, six is called the *minuend*, four the *subtrahend* and two the *difference*.

It is obvious that if $6+7=13$, then $13-6=7$ and $13-7=6$. The two inverses are alike. In the subtraction of numbers the minuend is usually placed above and the subtrahend below, thus

$$\begin{array}{r} 634 \\ 431 \\ \hline 203 \end{array}$$

Frequently, in the ordinary process of subtraction, it is necessary to subtract a larger digit from a smaller. For example, in

$$\begin{array}{r} 631 \\ 454 \\ \hline 177 \end{array}$$

we cannot very well take 4 from 1. We say $31=20+11$, and subtract our 4 from 11, which gives us 7. Of course, when we come to the next operation, we must not forget that we now have to consider 5 from 2 (not 3). So proceeding as before, we say $62=50+12$ and 5 from 12 leaves 7. That makes the last calculation 4 from 5, which is 1. Should we wish to prove our subtraction, we simply add together the subtrahend and the difference, in the present case 454 and 177. If this gives us 631 (the minuend), then we know our subtraction is correct.

Again we must bear in mind that speed and accuracy go hand in hand and try to obtain such a mastery of these operations as will enable us to perform them readily and without difficulty. Subtraction is preferably performed much in the way of making change in stores, where customers and salesmen subtract two or three numbers as quickly as one. The

last example (631—454) may be worked out as follows: 4 and how many make 11? Answer 7. Then 5 and the 1 to carry from the 11 make 6, and how many makes 13? Answer 7. Again, 4 and the 1 to carry from the 13 make 5 and how many make 6? Answer 1.

So when we have to take away two numbers from a third, we set the figures down as shown and proceed as follows: 8 and 3 are 11 and how many make 14? Answer 3. Then 1 (to carry from 14) and 1 and 4 make 6 and how many make 8? Answer 2. Then 2 and 6 are 8 and how many to make 12? Answer 4. This method of subtraction was introduced into this country at West Point and is called the Austrian Method.

$$\begin{array}{r} 1284 \\ 643 \\ 218 \\ \hline 423 \end{array}$$

6. Multiplication.—Multiplication is really only an extended form of addition. Instead of saying $2+2=4$, we may say twice $2=4$. Instead of $2+2+2=6$, we may say three times $2=6$. Again, 127 times 103, if done by addition, would require that we put down 127 as many as 103 times and add them together; or 103 as many as 127 times, which is the same thing. It is much simpler and quicker to do the operation by multiplication.

The sign of multiplication is $\times$ (times) placed between the numbers to be multiplied. Example: $8\times5=40$. 8 and 5 are called *factors* and 40 is the *product*. 8 is sometimes called the multiplicand and 5 the multiplier.

It is necessary in order to multiply readily and rapidly that we know by heart our multiplication table, at least up to and including 9. In the table given here we find at a glance the product of any two factors up to 9 Thus, after locating 8 in the left-hand column and 5 in the top row, we simply observe

where the "eight" row and the "five" column meet; here we find the product 40.

Multiplication Table

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	13	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

When we multiply two numbers, say 654×328, we are really performing three separate operations and adding the products together. We first multiply 654 by 8 and put down the product; then we multiply the same number by 20 and put that down, and finally we multiply it by 300 and put that down. Adding these factors together (5232+13080+196200) we get 214512. In practice we do the same thing somewhat more rapidly as follows:

$$\begin{array}{rl} 654 & \\ 328 & \\ \hline 5232 & = 654 \times 8 \\ 13080 & = 654 \times 20 \\ 196200 & = 654 \times 300 \\ \hline 214512 & \end{array}$$

7. The Decimal Point in Multiplication.—Numbers having decimals are multiplied just as tho they had none, but some care is necessary in determining where the decimal point is to be placed in the product. Take 6.54 and multiply it by 32.8. Ignoring for the moment the decimal points, we multiply the two numbers in the ordinary manner and get as product, as we did

with the foregoing example, 214512. But one of our factors had one decimal and the other two. How many decimals is the product to have in that case?

The common rule for placing the decimal point in a product of two factors is that the product will have as many decimals as the two factors have together, or more accurately stated: *The number of decimal places in any product is the sum of the decimal places in the members.* According to this rule, the product of our multiplication should have three decimals, hence it will be 214.512.

The decimal point is not a place. It is there only to point out the unit. Places are counted to the right and left of the unit, not to the right or left of the point. *If we multiply by a figure two places to the right or left of the unit's place, our result will have the point two places to the right or left of where it was in the number multiplied.* The multiplication of 6.54 by .08, where the 8 is two places to the right of the unit, puts the decimal point two places to the left of where it was in the number multiplied, thus: $6.54 \times .08 = .5232$. This rule will apply where the other cannot be used.

8. **Division.**—Just as subtraction is the inverse of addition, so division is the inverse of multiplication. Division involves finding how many times a certain number or quantity is contained in another. Thus, if $3 \times 4 = 12$, it is readily seen that 4 goes 3 times into 12, and that 3 goes 4 times into that number. The sign for division is $\div$ (meaning *divided by*), and is used as follows: $12 \div 4 = 3$. Here 12 is the *dividend*, 4 the *divisor* and 3 the *quotient.* When the divisor is not contained an exact number of times in the dividend, as in the case of $15 \div 7$, that which is left over is called

the *remainder*. In this case the remainder is 1. The remainder will, of course, always be less than the divisor.

In order to divide readily, we must know our multiplication tables. We quickly learn that $16 \div 2 = 8$ because we know that $2 \times 8 = 16$. Similarly, since we know that $9 \times 9 = 81$, we also know by that fact that $81 \div 9 = 9$.

The distinction sometimes made between short division and long division is an arbitrary one, since the process is exactly the same in all cases. In the example $1686 \div 7$, we notice that 7 is not contained in 1, the first figure, so we begin by saying 7 into 16 goes twice and leaves a remainder of 2. Alongside this remainder we place the next number in the dividend (8) making the number 28, and divide our 7 into it. This goes exactly 4 times without a remainder. The dividend has now only one figure left, namely 6. Into this the divisor 7 will not go but leaves a remainder of 6. The entire operation is seen in the example, in which the top line is the quotient, where the 2, representing the number of times 7 is contained in 16, is placed over the last figure of 16. *Places* in *quotient* and *dividend correspond*. For practical purposes we would call this quotient 241 instead of 240, the reason being that the remainder is very nearly as large as the divisor. In other words, had the dividend been 1687 instead of 1686, the number 7 would have gone exactly 241 times. As a general rule, where the remainder is one-half or more the amount of the divisor, the last number in the quotient is made one larger.

```
  240
7)1686
  14
  28
  28
   6
```

If we wish to make certain that our division is correct, we simply multiply the quotient by the divisor and add the remainder. This should equal the dividend. Thus $240 \times 7 + 6$ is found to equal 1686, which proves our division right.

Note: Mathematicians have agreed that in these operations multiplication and division shall be performed before addition and subtraction.

9. Decimal Point in Division.—Where dividend and divisor—either or both—have decimals, we proceed with our division just as tho they were whole numbers; but in writing the quotient, the place of the decimal point must be given careful attention. The following simple rule is commonly given: *The quotient has as many decimals as the dividend has more than the divisor.* Thus if the dividend has three decimal places and the divisor two, the quotient will have one. If dividend and divisor have the same number of decimal places, the quotient will have none. Should the divisor have more decimal places than the dividend, zeros are often added to the right of the latter so as to give it as many decimal places as the former.

So just as in multiplication we add the number of decimals, in division we subtract one from the other. As this rule, however, cannot be applied conveniently in many cases of measured numbers, we give the following method, which is always workable:

Suppose we have to divide 826.437 by 17.63. Evidently, the result will be the same if we multiply each number by 100 and divide 1763 into 82643.7, that is, move the decimal point far enough to the right to make the divisor a whole number and move the decimal point in the dividend just as many places and in the same direction. In the example, we will use a comma to denote the new position of the decimal point. The problem is worked out the same as the former from now on. The decimal point of the result always comes above the (,).

```
          46.9
17.63,)826.43,7
       705 2
       -----
       121 23
       105 78
       ------
        15 45 7
        15 86 7
        -------
```

In our quotient, we take the *nearest* last figure, as we did in short division.

10. Correct Measurement.—We cannot measure anything with complete exactness. We measure distance to the nearest mile, the nearest foot, the nearest inch.

Watches are standardized to the nearest thousandth part of an inch. Still the fact remains that exact measurement is impossible. If two pieces of metal are exactly the same length at one instant they will be different lengths a little later on account of change in temperature.

Suppose you measured the length of a table several times with a finely graduated rule and obtained nearly the same measure each time. In order to reduce the element of error as much as possible, you measure carefully, say ten times, and find the ten measurements to be as given herewith.

36.596
36.589
36.590
36.599
36.590
36.588
36.587
36.595
36.594
36.598
365.926

Adding these together and dividing by the number of them (10) gives 36.59 to the nearest hundredth of an inch. The last two figures may be ignored, since they are less than 50. The result 36.59, is called the *arithmetical mean* of the numbers given and is accepted as the most probable value of the length measured.

You will observe that in this instance we measured the table to three places of decimals, that is, we used five figures —*significant figures* as they are generally called. But we retain only four. It is usual, in making short computations, to retain one less figure than we measured to. In long computations, however, it is safer to drop the last two figures, counting half or more than half as a whole of the next place, bearing in mind that in practice it is the first figures that really count—the significant figures—rather than decimal places.

The table, as we have computed, is 36.59 (to be read, "thirty-six point five nine") inches long. Suppose we find in the same manner that the table is 12.83 inches wide to the nearest hundredth part of an inch. The area of the table top is then found by multiplying these two numbers and calling the result square inches. (The subject of area is one which is developed further in the chapters on geometry.)

11. Care in Computing Measurements.—The multiplication of 36.59 by 12.83 is worked out ordinarily in one of the two following ways:

36.59		36.59
12.83		12.83
365.9	.....	10977...
73.1	8....	29272....
29.2	72...	7318.....
1.0	977..	3659.....
469.4	497..	469.4497.. = 469. in whole numbers.

The two examples are the same except that they are performed in reverse order, making the columns incline in contrary direction. The dots indicate some of the missing figures, which in any place might be any number from 0 to 9, that is, any one of ten figures. The dots following each other indicate any number at all up to a hundred and three dots in a row say that we have one chance in one thousand of guessing the number indicated by them. This is a curious situation. How much is 7 and three numbers indicated by dots? It can be any number from 7 up to 34, according as the dots are 0's or 9's. The next two columns are not much better. In these circumstances the question naturally arises whether there was any justification for our putting down the figures 497.

The steps by what is called the *contracted method* are as follows:

36.59	36.5~~9~~	36.~~59~~	3~~6~~.~~59~~
12.83	~~1~~2.83	~~1~~2.83	~~1~~2.~~8~~3
365.9	365.9	365.9	365.9
	73.2	73.2	73.2
		29.2	29.2
			1.1
			469.4 = 469. in whole numbers

The digits of the multiplication are crossed off as used, and the figure last crossed off in the upper number

is used for carrying only, as otherwise the product would invade a place to the right of tenths about which we know nothing. Thus in the second step we say $2\times9=18$ and as 18 is nearer to 20 than 10 we carry 2. This method avoids writing the useless and deceiving figures to the right of our line in the first multiplication of this problem. David Rittenhouse, the famous American astronomer of Revolutionary times, employed this method as readily as Government computers do today. It is both shorter and more accurate than the old method, and so also is the problem next following.

```
          12.83
36.59,)469.5.,
       365.9
       -----
       103.6
        73.2
       -----
        30.4
        29.2
       -----
         1.2
         1.1
```

In division we cross off successively the last figures of our divisor instead of adding zeros. Thus the second division is by 365 and here again the 2×9 (the figure crossed off) gives us 2 to carry.

12. Divisibility.—There are some rules of exact division that should be noted, among them the following: *If a number is divisible by another number, any number of times the first number is also divisible by the second number.* For example, since 12 is divisible by 3, so is 60, which is 5×12, and 72 which is 6×12, etc. Again, *if two numbers are each divisible by a number, their sum and difference are also divisible by that number.* The numbers 12 and 60, for example, are both divisible by 3, hence their sum (72) and their difference (48) are also divisible by 3.

An *even* number is a number divisible by 2. Hence all numbers that end in 0, 2, 4, 6, 8 are even numbers.

Numbers which are not even are called *odd*. Thus 1, 3, 5, 7, etc., are odd numbers. A *composite* number is one that can be divided without a remainder. All even numbers except 2 are composite. So are many odd numbers, as for example 9, which is 3×3; 15, which is 3×5, etc. Numbers which are not composite are called *primes*.

13. Prime Numbers.—Eratosthenes, a citizen of ancient Greece, wrote on a piece of parchment the odd numbers 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25. Then starting with 3 he punched out every third number, as 9, 15, 21; and starting with 5, every fifth number (15, 25). He found that the remaining numbers were prime, since all the composite numbers had been crossed out of the above list. Eratosthenes punched his numbers out with the point of his cane, which left his parchment so full of holes that it came to be called "Eratosthenes' sieve". Such a table of primes is often of great practical use and should be made to include all primes below one hundred.

14. Rules of Exact Division.—Among other useful rules of exact division there are the following:

Any number whose last two figures are divisible by 4, *is itself divisible by* 4. Example: 536 is divisible by 4, since 36 is divisible by 4.

Any number whose last three figures are divisible by 8, *is divisible by* 8. Thus, 34,664 is divisible by 8 since 664 is divisible by 8.

Any number the sum of whose figures (*or digits*) is *divisible by* 9, *is itself divisible by* 9. The number 5,345,136 is divisible by 9 since adding the figures gives us 27, which is divisible by 9.

It also holds that *if the sum of the digits of a number is divisible by 3, so also is the number.* It is likewise clear that *a number ending in 5 or 0 is divisible by 5.*

Certain numbers are readily seen to be divisible by 11. Wherever we find the difference between the odd digits, (first, third, fifth, etc.) and the even digits, (second, fourth, sixth, etc.) to be either 0 or some multiple of 11, we can be sure that the number itself is divisible by 11. Thus the number 52,916,281 is divisible by 11 since the sum of its odd digits (28) less that of its even digits (6) leaves 22, which is a multiple of 11.

There are also rules for divisibility by 7, but they are rather cumbersome for general use, hence we omit them. We may note, however, that a number is divisible by 6 if it is even and if the sum of its digits is divisible by 3. Similarly, divisibility by 8 does not prevent divisibility by 9 or 5. The numbers 6 and 9, being composite numbers, are both divisible by 3, which is called their *common divisor.* The numbers 8 and 9 are also composite numbers, but they have no common divisor. They are called *relative primes.*

15. Greatest Common Factors.—Suppose we wish to divide a composite number into its prime factors. Take as an example 2145. Since this number ends in 5, we know at once that it is divisible by 5. This, therefore, is one prime factor. The quotient of the division by 5 is 429, and as $4+2+9=15$ we know that 429 is divisible by 3. Here is our second prime factor. Dividing 429 by 3, we obtain 143. Again applying a rule just learned, we find that the sum of the odd digits (1 and 3) is the even digit 4, hence

$$\begin{array}{r} 5)2145 \\ 3)429 \\ 11)143 \\ 13 \end{array}$$

we know that 143 is divisible by 11, the quotient being 13. The prime factors of 2145 are therefore 3, 5, 11, 13. By way of proof, multiply these numbers and you will get 2145. Had we been unable to recognize prime factors by the foregoing laws of division, we should have had to try in turn each of the primes from Eratosthenes' sieve.

We understand that common factors or divisors are factors which are common to two or more numbers. Thus 3 is a common factor of 12 and 15. It is also the *greatest common factor* (generally written G. C. F.) of these numbers. To find the greatest common factor of, say, 294, 378 and 462, we proceed on lines already indicated. We find that the number 2 is a common factor of these, hence we divide them by 2 as shown. Again the three quotients are divisible by 3, and the remainders in turn divisible by 7. There being no common factors of 7, 9 and 11, we then multiply the common factors found ($2\times3\times7$) which gives us 42 as the G. C. F. of 294, 378 and 462. The result may be verified by division.

2)	294	378	462
3)	147	189	231
7)	49	63	77
	7	9	11

16. Multiples.—We see at a glance that 12 is a multiple of 2, 3, 4 and 6. It is a *common multiple* of 2, 3, 4, 6—in fact, the *least common multiple* of these numbers. Evidently a common multiple of two or more numbers contains the prime factors of each of them. When such a multiple is a divisor of all the multiples of these numbers, we call it the least common multiple, generally written L. C. M.

Let us find the least common multiple of 60, 90 and 210.

$60 = 2 \times 2 \times 3 \times 5$
$90 = 2 \times 3 \times 5 \times 3$
$210 = 2 \times 3 \times 5 \times 7$

The least common multiple of 60, 90 and 210 must have the prime factors 2, 2, 3 and 5 in order to contain 60, it must have the additional prime factor 3 in order to contain 90, and the additional prime factor 7 in order to contain 210. Therefore, $2 \times 2 \times 3 \times 5 \times 3 \times 7$, or 1260 = L. C. M. of 60, 90, and 210.

17. Cancellation.—This is a method of shortening the work in computation. The principle is that multiplying one number and dividing another by the same number does not change their product. Thus $6 \times 20 = 12 \times 10$. Here we have multiplied 6 by 2 and divided 20 by 2. The result is the same—120.

Again, multiplying both the dividend and the divisor by the same number does not change the quotient; neither does dividing the dividend and the divisor by the same number.

Examples: (a) $12 \div 3 = 60 \div 15 = 4$. (b) $21 \div 9 = 7 \div 3$, etc. Based upon this fact we cancel or reject equal factors from dividend and divisor. Thus:

$$\frac{9 \times 27 \times 24 \times 40}{36 \times 9 \times 15 \times 2} = \frac{\cancel{9} \times \overset{3}{\cancel{27}} \times \overset{6}{\cancel{24}} \times \overset{4}{\cancel{40}}}{\underset{4}{\cancel{36}} \times \cancel{9} \times \underset{5}{\cancel{15}} \times \cancel{2}} = 24.$$

Here we proceed as follows: First we cancel the two 9's. Then we note that 27 in the dividend—upper row—and 36 in the divisor—lower row— are each divisible by 9, the former 3 and the latter 4 times. So we strike out 27 and 36 and write instead 3 and 4. Next we observe that the 4 in the divisor goes 6 times in 24 of the dividend; therefore 4 and 24 are cancelled and 6 is written above 24. We see also that 15 below is divisible by 3 above; consequently they also are cancelled and 5 is written under the 15. Then 5 is contained in 40 just 8 times and 2 in 8 goes

4 times. By this time, every number has been rejected from the divisor and all but 6 and 4 from the dividend. Multiplying these we have 24 as the answer.

For the sake of acquiring facility in cancelling, it will be well to work out the following:

$$\frac{9\times8\times18\times45}{36\times3\times90} = ? \qquad \frac{26\times18\times20}{39\times60\times 2} = ?$$

18. Casting Out Nines.—A simple way of checking the correctness of addition, subtraction, multiplication and division, and in fact for every exact computation, is called "casting out nines." Here is an example of this system as applied to addition:

	Excess
16	7
28	1
47	2
91	10

Having added 16+28+47 and obtaining 91 as the sum, we add the digits of the first number 1+6=7. There being no 9 to cast out, we place 7 in the position shown. Then 2+8=10, which is 1 more than 9. Casting out the 9 we have 1 left, which we place below the 7, as shown. Next 4+7=11, which is 2 more than 9. The 9+1 of the sum=10; so also do the 7, 1 and 2 of the excess column. If these results give the same excess over 9, we are reasonably certain that the answer is correct. The only possible error is that the result might be 19 instead of 91.

	Excess
269734	4
528631	7
429982	7
534287	2
886321	1
2648955	3

A longer problem is equally simple, as the second example shows. As the excess over 9 is 3 in either case, the result is probably correct.

In proving subtraction and the other computations, we perform the same operation with the excess as is done with the original

numbers. When we have to subtract a number from a smaller number, we add 9 to the smaller number before making the subtraction.

	Excess
84	3
23	5
61	7

Here is a simple example of subtraction, in which case before subtracting the 5 we added 9 to the 3=12. Then 12−5=7.

The example in multiplication easily explains itself.

	Excess
76	4
47	2
532	
304	
3572	8

The following operation in division is equally simple:

Excess			Excess
	80		8
3	48)3847		4
	384		
	7		7

Inasmuch as the dividend= the divisor times the quotient+ the remainder, the excess dividend will equal the excess divisor times the excess quotient plus the excess remainder, that is 3×8+7 or excess 31=4.

Note: This method obviously will not check an error caused by a reversal of figures, as for example 587 instead of 785.

REVIEW.

1. Describe how the early ancients used their fingers in computation.

2. What does the zero represent? How did computers manage without it before it first came into use?

3. Give a short account of the value of the decimal point.

4. What is the relationship between addition and multiplication?

5. Mention the terms employed in subtraction and division.

6. When multiplying numbers having decimal points how do you determine the position of the decimal point in the result? Also in division?

7. Why is accurate measurement so difficult to obtain? What is the best method of reducing the element of error?

8. What are prime numbers? The greatest common factor? The least common multiple?

CHAPTER II

FRACTIONS

1. Proper and Improper Fractions.—If a pie were equally divided among four people, each would get $\frac{1}{4}$ (read one fourth) of the pie. Three of the people would get $\frac{3}{4}$ of the pie. If seven people were each to receive a $\frac{1}{4}$ pie they would need one whole pie and $\frac{3}{4}$ of another one, that is, $1\frac{3}{4}$ or $\frac{7}{4}$ of a pie. Thus $\frac{3}{4}$ would mean 3 of the 4 equal parts into which the pie or any other thing was divided. And $\frac{7}{4}$ would represent 7 such parts.

Numbers like $\frac{3}{4}$ were introduced into our number system long after whole numbers came into use. Such numbers as $\frac{1}{4}$, $\frac{3}{4}$, $\frac{2}{3}$ are called *fractions*. If the *numerator*, or upper figure, is less than the *denominator*, or lower figure, as in the cases just mentioned, we have *proper fractions*. When the numerator is greater than or equal to the denominator we have *improper fractions*. Thus $\frac{7}{4}$ and $\frac{4}{4}$ are both improper fractions. The numerator and denominator are often called the *terms* of the fractions. Improper fractions are readily reduced to whole or mixed numbers. Thus $\frac{4}{4}=1$; so $\frac{12}{4}=3$; also $\frac{7}{4}=1\frac{3}{4}$ and $\frac{13}{5}=2\frac{3}{5}$. In the fraction $\frac{3}{5}$, the denominator shows into how many equal parts the unit is divided, and the numerator, how many of these parts are brought into reckoning, or we may understand that 3 is to be divided into 5 equal parts. Other shades of meaning will come before us later.

The following three proper fractions have the same denominator (5) and are arranged according to size, the least first: $\frac{2}{5}, \frac{3}{5}, \frac{4}{5}$.

The following three improper fractions have the same numerator (6) and are arranged according to size, the greatest first: $\frac{6}{3}, \frac{6}{4}, \frac{6}{5}$.

Supply at sight the missing terms in the following:

$$\frac{2}{3} = \frac{?}{24} \qquad \frac{3}{4} = \frac{15}{?} \qquad \frac{21}{24} = \frac{?}{8}$$

$$\frac{16}{24} = \frac{2}{?} \qquad 1\tfrac{1}{2} = \frac{?}{8} \qquad \frac{210}{336} = \frac{?}{8}$$

Suggestion: As the new denominator (24) is eight times the old denominator (3), so the numerator must be eight times the old numerator or 16. As 15 is 5×3, so the denominator must be 5×4 or 20. And so on.

A fraction is said to be in its lowest terms when its numerator and denominator have no common divisor.

Thus in the example $\frac{210}{336}$ has been reduced by successive cancellation to its lowest terms ($\frac{5}{8}$).

$$\begin{array}{r} \cancel{210} \\ \hline \cancel{336} \end{array} = \frac{5}{8}$$

$$\begin{array}{r} 5 \\ \cancel{35} \\ \cancel{105} \\ \cancel{210} \\ \cancel{168} \\ \cancel{56} \\ 8 \end{array}$$

No fraction is considered simplified until it has been reduced to its lowest terms.

For practice reduce to their lowest terms $\frac{12}{15}, \frac{42}{63}, \frac{144}{288}, \frac{274}{300}$.

2. Mixed Numbers.—As in the whole number 5, there are $\frac{20}{4}$, so in $5\frac{3}{4}$ there are necessarily $\frac{23}{4}$.

The reduction of mixed numbers to improper fractions is performed by multiplying the whole number by the

denominator and adding the numerator, the sum being the new numerator. The denominator remains as before. Thus $2\frac{1}{2} = \frac{2 \times 2 + 1}{2} = \frac{5}{2}$.

Similarly, $3\frac{1}{3} = \frac{10}{3}$; $55\frac{1}{3} = \frac{166}{3}$; $66\frac{2}{3} = \frac{200}{3}$, etc.

Reducing improper fractions to mixed numbers is the reverse process to that just considered. That is, we divide the numerator by the denominator and take the remainder as the numerator of the new fraction.

Thus $\frac{100}{3} = 100 \div 3 = 33\frac{1}{3}$; $\frac{513}{4} = 128\frac{1}{4}$.

3. Common Denominator.—From what we have already learned, it is evident that $\frac{3}{5} = \frac{9}{15}$ and that $\frac{2}{3} = \frac{10}{15}$. These two fractions have been reduced to a *common denominator* and we can now say that $\frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15}$ or $1\frac{4}{15}$ as readily as we can say \$10+\$9=\$19, because we are adding quantities of the same kind. It is, then, evident that fractions, to be added or subtracted, must be reduced to a common denominator, which is a common multiple of the denominators. In practice, the *least common denominator* (*L. C. D.*) is generally used, and this, it will be seen, is the L. C. M. (least common multiple) of the denominators.

Examples: $\frac{7}{8} + \frac{3}{4} + \frac{1}{2} = \frac{7 + 6 + 4}{8} = \frac{17}{8} = 2\frac{1}{8}$.

$$\begin{aligned} 124\tfrac{7}{8} &= 124\tfrac{28}{32} \\ 346\tfrac{5}{16} &= 346\tfrac{10}{32} \\ 452\tfrac{7}{32} &= 452\tfrac{7}{32} \\ \text{Sum} &= 923\tfrac{13}{32} \end{aligned}$$

For the sake of brevity, it is customary to write the L. C. D. only once and not under each numerator.

Thus $\frac{13}{24} - \frac{13}{40} = \frac{65 - 39}{120} = \frac{26}{120} = \frac{13}{60}$.

From the foregoing it will be easily understood how subtraction of mixed numbers is done. Suppose we wish to subtract $98\frac{3}{4}$ from $124\frac{3}{8}$.

$124\frac{3}{8} = 123\frac{11}{8}$

$98\frac{3}{4} = 98\frac{6}{8}$

Diff. $= 25\frac{5}{8}$

Since $\frac{3}{4}$ is more than $\frac{3}{8}$, we take one from 124 and add it to $\frac{3}{8}$ making $124\frac{3}{8} = 123\frac{11}{8}$. From this we subtract $98\frac{6}{8}$ and obtain $25\frac{5}{8}$.

4. Multiplication of Fractions.—It is self-evident that $3 \times \frac{2}{7} = \frac{2}{7} + \frac{2}{7} + \frac{2}{7} = \frac{6}{7}$. We could also state it this way: $\frac{3 \times 2}{7} = \frac{6}{7}$. Also $9 \times \frac{5}{48} = \frac{\overset{3}{\cancel{9}} \times 5}{\underset{16}{\cancel{48}}} = \frac{15}{16}$.

It is easily seen that $\frac{2}{3}$ of $\frac{4}{5} = \frac{2}{3} \times \frac{4}{5}$ is the same as $2 \times \frac{1}{3}$ of $\frac{4}{5}$. Also $2 \times \frac{1}{3}$ of $\frac{4}{5} = 2 \times \frac{4}{15} = \frac{8}{15}$. This could have been put in the following form: $\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$, that is, multiplying the two numerators and the two denominators as shown.

So also $\frac{25}{24} \times \frac{16}{15} = \frac{25 \times 16}{24 \times 15} =$ (after cancellation) $\frac{10}{9} = 1\frac{1}{9}$.

Similarly, $\frac{125}{2} \times \frac{3}{75} = \frac{125 \times 3}{2 \times 75} = \frac{5}{2} = 2\frac{1}{2}$.

Again $\frac{63}{100} \times \frac{75}{54} = \frac{63 \times 75}{100 \times 54} = \frac{7}{8}$.

To multiply $456\frac{5}{6}$ by 25 is simply to multiply 456 by 25 = (11,400) and add $25 \times \frac{5}{6} = (\frac{125}{6} = 20\frac{5}{6})$ totaling $11{,}420\frac{5}{6}$.

To multiply $127\frac{1}{4}$ by $46\frac{2}{3}$ is equally simple. First multiply

	127 by 46	$= 5842$
Next	$\frac{1}{4}$ by 46	$= 11\frac{1}{2}$
Then	127 by $\frac{2}{3}$	$= 84\frac{2}{3}$
Finally	$\frac{1}{4}$ by $\frac{2}{3}$	$= \frac{1}{6}$
Which when added together make		$5938\frac{1}{3}$

5. Division of Fractions.—Since $\frac{27}{50}$ means 27 of the 50 equal parts into which a unit has been divided, $\frac{27}{50} \div 3$ evidently equals $\frac{9}{50}$, since 3 goes into 27 nine times. When we wish to divide $\frac{27}{50}$ by 2, however, we find that 27 is not divisible in whole numbers by 2. In that case our course is as follows: As $\frac{1}{2}$ of $\frac{1}{50}$ equals $\frac{1}{100}$, therefore $\frac{1}{2}$ of $\frac{27}{50}$ is $\frac{27}{100}$. In other words, we divide fractions either by dividing the numerator or by multiplying the denominator, which give the same result, since *dividing the numerator divides the number of parts, while multiplying the denominator reduces the size of each part.*

In dividing $\frac{27}{50}$ by 45, we find it more convenient to multiply the denominator (50) by 45 than to divide the numerator (27), hence $\frac{27}{50} \div 45 = \frac{27}{50 \times 45} =$ (after cancellation) $\frac{3}{250}$.

In dividing, say, 5 by $\frac{2}{3}$, we note that $\frac{1}{3}$ is contained in 1 three times, and therefore in 5 fifteen times. Two-thirds, then, is contained in 5 only half as many times. Thus $5 \div \frac{2}{3} = \frac{5 \times 3}{2}$. We say in practice, "Invert the divisor and proceed as in multiplication." We have evidently inverted the divisor $\frac{2}{3}$, for we have multiplied 5 by $\frac{3}{2}$. The result of $5 \div \frac{2}{3}$ is therefore $\frac{15}{2} = 7\frac{1}{2}$.

$$\text{Accordingly } \frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}.$$

$$2 \div 1\frac{1}{3} = 2 \div \frac{4}{3} = 2 \times \frac{3}{4} = \frac{6}{4} = 1\frac{1}{2}.$$

$$2 \div 7\frac{1}{2} = 2 \div \frac{15}{2} = 2 \times \frac{2}{15} = \frac{4}{15}.$$

Suppose we wish to know the value of $\frac{4\frac{3}{4}}{47\frac{1}{2}}$. We multiply both numerator and denominator by 4, the common denominator of the two fractions. This gives us in the numerator 19, and in the denominator 190; hence our fraction is $\frac{19}{190} = \frac{1}{10}$.

Another example: $\frac{12\frac{1}{2}}{\frac{3}{8}} = \frac{100}{3} = 33\frac{1}{3}$. Here we multiply both numerator and denominator by 8.

When we have to solve a problem like this: $\frac{(\frac{2}{3} + \frac{3}{4}) \times (\frac{3}{4} - \frac{2}{3})}{(\frac{9}{10} - \frac{4}{9}) \div (\frac{2}{3} + \frac{3}{4})}$

we restate it thus: $\frac{(\frac{2}{3} + \frac{3}{4}) \times (\frac{3}{4} - \frac{2}{3}) \times (\frac{2}{3} + \frac{3}{4})}{(\frac{9}{10} - \frac{4}{9})}$. Our reason for doing so is that dividing the denominator is the same as multiplying the numerator. All we have to do then is to simplify these parentheses and complete the work as already shown.

In attempting to simplify the following: $\frac{2\frac{1}{2} \times 25 \times \frac{1}{2} \times 37\frac{1}{2}}{1\frac{1}{4} \times 12\frac{1}{2} \times 12\frac{1}{2} \times \frac{1}{8}}$,

we do well to note that every number below the line is divisible into a number above the line. After cancellation the fraction

will look like this: $\frac{2 \times 2 \times 4 \times 3}{1 \times 1 \times 1 \times 1} = \frac{48}{1} = 48.$

As a useful exercise, check the following for accuracy by adding

both in columns and in rows:
$$\begin{array}{r} \frac{5}{6} + \frac{7}{8} + \frac{2}{3} = ? \\ \frac{17}{24} + \frac{19}{24} + \frac{2}{3} = ? \\ \hline ? + ? + ? = ? \end{array}$$

6. Denominate Numbers.—Selecting a number at random, say 361, we know, but often forget from disuse, that this stands for 3 hundred 6 tens and 1 unit. The radix, or fundamental unit, is ten. But the quantity 3 pecks, 6 quarts, 1 pint seems different only because our regular counting method is fixed. When, therefore, in a number or quantity the different places are named, we have grown to call such numbers *denominate* or *named numbers.* For example, 4 miles, 2 years, 2 tons are all denominate numbers. So are 4 meters, 2 decimeters, 5 centimeters, etc., with this difference, that the latter group belongs to the metric system, where we come back to the single radix, ten, and the compound numbers can be written 4.25 meters or 42.5 decimeters or 425. centimeters, as we may choose.

7. Tables of Denominate Numbers.—A knowledge of the tables of denominate numbers is useful and in some cases indispensable. The following tables are those more generally used in every-day work. They should be memorized.

1. LINEAR MEASURE

12	Inches (in.)	= 1	Foot . . . . . ft.
3	Feet	= 1	Yard . . . . . yd.
$5\frac{1}{2}$	Yards or $16\frac{1}{2}$ Feet	= 1	Rod . . . . . rd.
320	Rods	= 1	Mile . . . . . mi.

2. SQUARE MEASURE

144	Square Inches (sq. in.)	= 1	Square Foot . . sq. ft.
9	Square Feet	= 1	Square Yard . . sq. yd.
$30\frac{1}{4}$	Square Yards	= 1	Square Rod . . . sq. rd.
160	Square Rods	= 1	Acre . . . . . A.
640	Acres	= 1	Square Mile . . . sq. mi.

3. CUBIC MEASURE

1728	Cubic Inches (cu. in.)	= 1	Cubic Foot . . . cu. ft.
27	Cubic Feet	= 1	Cubic Yard . . . cu. yd.
128	Cubic Feet	= 1	Cord . . . . . C.

4. LIQUID MEASURE

4	Gills (gi.)	= 1	Pint . . . . . pt.
2	Pints	= 1	Quart . . . . . qt.
4	Quarts	= 1	Gallon . . . . . gal.

5. DRY MEASURE

2	Pints (pt.)	= 1	Quart . . . . . qt.
8	Quarts	= 1	Peck . . . . . pk.
4	Pecks	= 1	Bushel . . . . . bu.

6. AVOIRDUPOIS WEIGHT

16	Ounces (oz.)	= 1	Pound . . . . . lb.
100	Pounds	= 1	Hundredweight . cwt.
20	Hundredweight	= 1	Ton . . . . . T.
2250	Pounds	= 1	Long Ton.

7. MEASURE OF TIME

60	Seconds (sec.)	=1	Minute.....min.
60	Minutes	=1	Hour.....hr.
24	Hours	=1	Day.....da.
7	Days	=1	Week.....wk.
365	Days	=1	Year.....yr.
366	Days	=1	Leap Year....yr.
100	Years	=1	Century.....cen.

8. CIRCULAR OR ANGULAR MEASURE

60	Seconds (″)	=1	Minute.....(′)
60	Minutes	=1	Degree.....(°)
360	Degrees	=1	Circumference Cir.

8. Facts About the Tables.—The first three of the tables —Linear (or Long), Square and Cubic Measures—are closely allied, the second and third being merely progressive developments of the first. We must, however, understand them in their proper order. A comparison of Linear and Square Measures might arouse an inquiry as to how it is that since 12 inches = 1 foot, 12 square inches do not equal 1 square foot. We quickly learn why, however, when we understand the relation between length and surface and volume.

Suppose we take a cube measuring 1 inch each way. The *surface* of each of the six faces of the cube is 1 sq. in. and the entire contents of the cube, the *volume*, is 1 cu. in. If twelve of these blocks were placed in succession in a straight row, they would extend a *length* of 1 foot, and they would cover a space of 12 sq. in. If, following this, twelve rows be placed together, side by side, there would be a square array of $12 \times 12 = 144$ blocks whose combined upper (or lower) surfaces would be 144 sq. in. Should eleven more such layers

be then laid on top of the bottom layer, making twelve layers in all, we would have a large block containing $12 \times 12 \times 12 = 12 \times 144 = 1728$ cu. inches.

In the illustration—Figure 1—we have a block containing three layers of nine smaller blocks or 27 blocks in all. If we assume that each small block contains 1 cu. ft., each layer, then, contains 9 cu. feet, and the entire block contains 3×9 cu. ft. = 27 cu. ft. = 1 cu. yd.

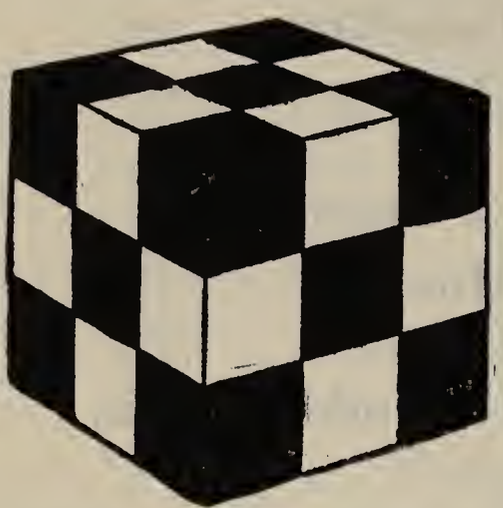

Fig. 1.

Careful standards of all measures are kept by the United States Government. There are more kinds of bushels in America than there are states, but the Government standard bushel contains 2150.42 cubic inches. Again, the gallon (Liquid Measure), which contains 231 cubic inches (Cubic Measure) will hold about $62\frac{1}{2}$ pounds of water (Avoirdupois Measure).

9. Exercises in Denominate Numbers.—The longitude of Philadelphia is 75° 09′ 23.4″ west of Greenwich. Suppose we wish to know what time it is in Greenwich when it is noon in Philadelphia.

As there are 360° in the circumference of the earth and as the earth turns once every 24 hours, it passes over $\frac{360°}{24} = 15°$ each hour. Hence, we divide degrees by 15 giving the number of hours, and as there are the same number of minutes and seconds in an hour as there are minutes and seconds in a degree, the same relation holds. Therefore, the answer is found in the following computation: $\frac{75°\ 9'\ 23.4''}{15} = 5$ hr. 0 min. 37.56 sec. P.M., (as the sun crosses the Greenwich meridian first.)

Two places differ in longitude by 85° 16′ 45″. By how much do they differ in time?

We work the problem as follows: We know from the preceding example that the earth passes over 15° each hour, therefore dividing the longitude by 15 will give us the difference in time between the two places. Dividing 85 by 15 we obtain a quotient of 5 with 10 over. This 10 we might multiply by 60 and call it minutes, but as we should then have to divide by 15, we instead divide 60 by 15=4 and say 4 times 10=40, and as 15 goes into 16 once, we have 40+1=41 minutes of time. Multiplying the 1′ left by 4 and adding the 3 which we got by dividing 45 by 15, we have 7 seconds of time.

$$\text{Thus } \frac{85^\circ\ 16'\ 45''}{15} = 5 \text{ hr. } 41 \text{ min. } 7 \text{ sec.} \quad \text{Answer.}$$

Suppose the sum of two angles to be 90°, one of them being 63° 12′ 15″. What is the other angle. ?

It will simplify matters if we say that 90°=89° 59′ 60″ and work the problem out by subtraction as shown:

$$\begin{array}{rrr} 89^\circ & 59' & 60'' \\ 63 & 12 & 15 \\ \hline 26^\circ & 47' & 45'' \end{array} \quad \text{Answer.}$$

The angles of a triangle are together equal to 180°. Two of them are 48° 15′ 32″ and 59° 54′ 15″ respectively. What is the third?

We could add the two together and subtract from 180°, but a better way is to place the numbers as in the form. We say 15″+32″=47″ and how many make 60″? Answer 13″. There is one minute to carry from the 60″, hence 1′+54′+15′=70′, and how many are 120′? Answer, 50′. There is now 2 to carry as 120′=2°. Then 2°+48°+59°=109°, and how many make 180°. Answer, 71°. Complete answer: 71° 50′ 13″.

Form

$$\begin{array}{rrr} 180^\circ & 00' & 00'' \\ 48^\circ & 15' & 32'' \\ 59^\circ & 54' & 15'' \\ \hline 71^\circ & 50' & 13'' \end{array}$$

Reduce 15 miles per hour to feet per second.

$$\begin{array}{r} 22 \\ \not{88} \\ \hline \not{60}\times\not{60} \\ \not{4} \end{array} = 22$$

Since there are 5280 feet in a mile and 60 times 60 seconds in an hour, the answer is 22 feet per second.

A sprinter who runs 100 yards in 10 seconds is running at what rate in miles per hour?

There are 1760 yards in a mile. One hundred yards in 10 seconds is at the rate of 10 yards per second. The work is as in the form.

$$\begin{array}{r} 15\ \ 15 \\ \cancel{10}\times\cancel{60}\times\cancel{60} \\ \hline \cancel{1760} \\ \cancel{44} \\ 11 \end{array} = \frac{225}{11} = 20\frac{5}{11} \text{ miles per hour.}$$

A policeman measures off 600 yards of road and notices that Mr. Smith goes over it in his car in less than a minute. Has Mr. Smith exceeded the speed limit of 20 miles per hour? Compare this with the previous problem and work it out by way of exercise.

10. Operations in Compound Numbers.—Our English forefathers had a great many tables of compound numbers, as they are sometimes called. They added and subtracted, multiplied and divided such numbers as 13° 10′ 11″ by 4° 15′ 6″ in practically the same way as the Babylonians did 4,000 years before them. Nowadays we do less of that sort of thing, but if we wished to know what part of 13° 10′ 11″ is 4° 15′ 6″, we would reduce both to seconds and then compare, as follows:

$$\begin{array}{r} 13^\circ \\ 60 \\ \hline 780' \\ 10' \\ \hline 790' \\ 60 \\ \hline 47400'' \\ 11'' \\ \hline 47411'' \end{array} \qquad \begin{array}{r} 4^\circ \\ 60 \\ \hline 240' \\ 15' \\ \hline 255' \\ 60 \\ \hline 15300'' \\ 6'' \\ \hline 15306'' \end{array}, \qquad \frac{15306}{47411} \text{ Answer.}$$

11. The Metric System.—In 1799, France adopted the metric system. It was intended to make a *meter* equal

to $\frac{1}{10,000,000}$ part of the distance from the earth's equator to its pole, but a slight error was made in the original computation. The standard is a platinum bar one *meter* (about 39.37079 inches; approximately 1.1 yards) in length, at 4° Centigrade, kept in the archives of the International Bureau of Weights and Measures, Paris. This system of measures is now in general use in every country in the world with the exception of the United States and a part of the British Empire, and both Great Britain and the United States have legalized it. When used at all, it should not be in connection with any other system. The Greek prefixes, deka-, hekto-, kilo-, myria-, meaning 10, 100, 1000, 10,000, indicate multiples of the unit. The Latin prefixes, deci-, centi-, milli-, meaning $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, indicate decimal divisions of the unit.

TABLE OF LENGTH

The table of length is given herewith, the more important units being indicated by italics.

10 *Millimeters* (mm.)	=1	*Centimeter*	(cm.)
10 Centimeters	=1	Decimeter	(dm.)
10 Decimeters	=1	*Meter*	(m.)
10 Meters	=1	Dekameter	(Dm.)
10 Dekameters	=1	Hektometer	(Hm.)
10 Hektometers	=1	*Kilometer*	(Km.)
10 Kilometers	=1	Myriameter	(Mm.)

TABLE OF SQUARE MEASURE

The table of square measure starts with

100 Square Millimeters	=1	Square Centimeter
100 Square Centimeters	=1	Square Decimeter, etc.

The important units are the squares of those of the table of length. Sometimes 100 sq. meters is called an *are*.

TABLE OF CUBIC MEASURE

1000 Cubic Millimeters = 1 *Cubic Centimeter* (cu. cm.)
1000 Cubic Centimeters = 1 Cubic Decimeter (cu. dm.)
1000 Cubic Decimeters = 1 *Cubic Meter* (cu. m.)

In measuring wood, the cubic meter is called a *stere* (s.). It is subdivided by 10's. The cubic decimeter is called a *liter* and has subdivisions and multiples of 10, as in the first table. When water is at its greatest density, 4° Centigrade, a *cubic decimeter*, that is, a *liter*, weighs a *kilogram* (kg.) = 1000 grams = 1000 cubic centimeters.

It gives us a better idea of these measures if we know that a centimeter is about equal to the breadth of one's little finger nail, that a decimeter is about the length of the forefinger, and a meter about the length from the tip of one ear to the tip of the opposite middle finger when the arm is extended. A liter is about $\frac{1}{16}$ more than a liquid quart and between a liquid and a dry quart; a kilometer is about $\frac{5}{8}$ of a mile, a hektare is about $2\frac{1}{2}$ acres, a kilogram is about 2.2 pounds, a metric ton is about a long ton, a stere is little more than a quarter of a cord.

REVIEW.

1. How are the upper and lower figures in fractions named? What are proper and improper fractions?

2. How do you divide fractions?

3. What are denominate numbers?

4. Give a brief explanation of the relation between length, surface and volume.

5. Illustrate the modern method of multiplying the following compound numbers: 19° 23′ 11″ by 7° 52′ 37″.

6. What do the prefixes deka— and kilo—, and the suffixes deci— and milli—, indicate in the Metric System?

CHAPTER III

PERCENTAGE AND APPLICATION

1. Percentage.—Per cent, from the Latin *per centum*, means *on one hundred*. The sign of per cent is %. As the legal interest in this country is generally 6% per annum, one pays \$6 for the use of \$100 for one year. It follows, therefore, that 6% of a number equals $\frac{6}{100}$ or .06 of that number. Six per cent of \$250 is .06 of \$250 or $6 \times \$2.50$ or $2\frac{1}{2} \times \$6 = \$15$. The \$250 is called the *base;* 6 is the *rate per cent* or, more commonly, the *rate*, and \$15 is the *per centage*. Hence $25\% = .25 = \frac{1}{4}$ of a number. Also $\frac{1}{2} = .50$ or 50%.

If a grocer buys bacon at 26 cents a pound and sells it at 52 cents a pound, he sells it for twice the cost. He therefore sells it for 200% of the cost. He gains 100%. If a ball team has lost five games and won five games, it has won 50% of its games. If another team has won 6 out of 10 games, it has won 60% of its games and lost 40%.

What per cent of a number is $\frac{1}{3}$ of it? Answer: $\frac{1}{3}$ of $100\% = 33\frac{1}{3}\%$.

What fraction of a number is $12\frac{1}{2}\%$ of it? Answer: $\frac{12\frac{1}{2}}{100} = \frac{1}{8}$.

A man owes \$3485 and settles with his creditors on a basis of $37\frac{1}{2}\%$. What do his creditors receive?

As $37\frac{1}{2}\%$ is \$37.50 on each \$100, the amount the creditors are paid will be $\frac{37.50}{100}$ of \$3485 or .375 of \$3485. or $\frac{3}{8}$ of \$3485. Multiplying \$3485 by $\frac{3}{8}$ will give the result $\frac{3485 \times 3}{8} = \$1306\frac{7}{8} = \$1306.875$. It is better, in general, however, to work the problem out in decimals, as shown:

\$3485
.375
17425
24395
10455
\$1306.875

When we know the percentage and the rate per cent, we can find the base or sum by dividing the percentage by the percentage on one dollar. For example, of what sum is \$74.76 just 42%? Dividing 74.76 by .42 gives us \$178 as answer.

Similarly, \$240.20 is $66\frac{2}{3}\%$ of what sum?

$66\frac{2}{3}\%$ or $\frac{2}{3}$ of sum = \$240.20

Then $\frac{1}{3}$ of sum = 120.10

Therefore the sum $= 3 \times 120.10 = \$360.30$ Answer:

When we have the base and the percentage we readily find the rate per cent by multiplying the percentage by 100 and dividing by the base. As for example, the rate per cent when base is 300 and percentage is 18 is $\frac{18 \times 100}{300} = 6\%$. Sometimes, the problem permits of a simpler method, as for instance when we wish to find what per cent \$12.50 is of \$200. In this case we readily see that \$12.50 is the same per cent of \$200 as \$6.25 is of \$100, that is 6.25%, the answer.

What per cent of $\frac{4}{5}$ is $\frac{3}{4}$? We have $\frac{\frac{3}{4}}{\frac{4}{5}} = \frac{15}{16}$ (division of fractions) and $\frac{15}{16}$ of $100\% = 93\frac{3}{4}\%$.

What per cent of \$56.875 is \$3.00?

$$
\begin{array}{r}
.05275 = 5.27\% \\
56.875\overline{)3.000{,}00} \\
2\ 84375 \\
\hline
15625 \\
11375 \\
\hline
4250 \\
3981 \\
\hline
269 \\
284 \\
\hline
\end{array}
$$

The method in this example is always applicable, but the nature of the problem will often suggest the simpler methods explained previously.

2. Commercial Discount.—Commercial discount is any deduction made from the *list price*, the *time price* or the *marked price* of goods; the *net price* is the remainder.

If but one discount is made on any price, a certain per cent of that price is deducted; if two or more discounts are made, the first discount is reckoned on the price, the second on the remainder, the third on the next remainder and so on.

Thus, if an article is marked \$0.80 and sold at a discount of 25%, the discount is 25% of 0.80, or \$0.20 and the selling price is \$0.60. If it is sold at a discount of 25% and 10%, the second discount is 10% of \$0.60, or \$0.06, and the total discount is \$0.26. The selling price after the discounts is \$0.54.

Also, the net cost of a bill of goods listed at \$240 and bought at 20% and $12\frac{1}{2}\%$ off is found by first deducting 20%, or $\frac{1}{5}$, of \$240. Therefore \$48 is the first discount. Deducting this from the list price gives us \$192. The second discount, $12\frac{1}{2}\%$ of \$192 = \$24. Hence the net cost is \$168.

Business men generally convert two successive rates of discount into an equivalent single rate before they reckon the discount. They use the following rule:

A single rate of discount, equivalent to two successive

rates, equals their sum diminished by $\frac{1}{100}$ *of their product.*

Thus, a single rate of discount equivalent to 20% and 10% would be 20+10−2=28%. Prove this by deducting the discounts separately, as follows: 20% deducted from 100%=80%, 10% of 80%=8%, 20%+8%=28%.

In the case of three successive rates, we proceed according to the above rule and find the single rate equivalent to the first two, and thereupon combine this with the third. Thus, suppose the rates of discount were 20%, 10% and 5%, and the list price $100. By proceeding as above, we find the single rate for the first two to be 28%. Combining this rate with the third (5%), we find the single rate to be 33−1.40=31.60. In the present case the net amount would therefore be $100−$31.60=$68.40.

What single rate of discount is equivalent to 40%, 20% and 10%? Answer 56.80%.

3. Profit and Loss.—The *cost* of an article is the amount paid for it. The *selling price* is the amount received for it. The *profit* is the amount with which the selling price exceeds the cost and the expenses. The *loss* equals the amount with which the cost and expenses exceed the selling price. When there are no expenses, the profit equals the selling price less the cost, and the loss equals the cost less the selling price.

Thus if a dealer buys shoes at $2.50 a pair and sells them at a profit of 20%, it is evident that the selling price will be $2.50+.50=$3.00.

If coffee sold at 40 cents a pound yields a profit of 25% on the cost price, the latter must have been 32 cents, since 40 cents=$\frac{5}{4}$ of cost, hence $\frac{1}{4}$ or 25%=8 cents.

If an article was bought for \$40 and sold for \$55 the gain per cent is found as shown:

Gain	= \$15.
Cost	= 40.
Selling Price	= \$55.

It is then merely a matter of finding what part of the cost (\$40), the gain (\$15) is. This is $\frac{15}{40} = \frac{3}{8}$ of $100 = 37\frac{1}{2}\%$. Answer.

4. Commission.—A large amount of business of all kinds is transacted by agents for the purchasing or selling parties, who are called the *principals.*

An agent who receives into his possession the property of the principal, and transacts in his own name the business relating to such property, is called a *commission merchant.* An agent who does business in the name of his principal, and without receiving the property of the principal into his possession, is called a *broker.*

Commission or *brokerage* is the sum charged by an agent for transacting business for a principal, and is usually a certain per cent of the amount involved in the transaction.

Suppose an agent sells merchandise to the amount of \$3450. He pays storage and other expenses totaling \$35. What are the net proceeds for the principal, if the agent deducts a commission of 1.5%? The answer is found by simply adding together the expenses (\$35) and the commission ($1\frac{1}{2}\%$ of \$3450 = \$51.75), the sum of which is \$86.75, and deduct this amount from the selling price, leaving as net proceeds for the principal \$3363.25.

5. Simple Interest.—Interest is money paid for the use of money, called the *principal.* When the interest

is computed on the original principal only, it is called *simple interest.*

The time for which interest is paid is usually reckoned in years and days when it exceeds a year, and in days only when it is less than a year; the exact number of days being counted. Settlements of interest accounts are usually made at intervals of not more than a year, so that in practice it seldom becomes necessary to reckon interest on a sum of money for a time greater than a year. Interest for a time less than a year is usually reckoned on a basis of 360 days to a year; that is, 12 months of 30 days each, in which case the interest is called *common interest.* Interest for a time less than a year reckoned on a basis of 365 days to the year is called *exact interest.*

The following table shows the number of days from any day of one month to the same day of any other month in the same year, except when Feb. 29 intervenes, in which case one day is added:

If therefore we wish to ascertain the number of days from April 15 to September 23, we find, by referring to the table opposite April and under September, that the time from April 15 to September 15 is 153 days. From September 15 to 23 is 8 days, which must be added to the 153. The answer is therefore 161 days.

Suppose we are to find the number of days from April 15 to September 6. The table shows that April 15 to September 15 is 153 days. September 6, however, is 9 days before September 15, hence we subtract 9 days from 153 days, which gives 144 days as the correct time.

Again, suppose we wish to know how many days there are between January 6 and March 6 in 1920 or any other leap year. The table gives 59 days; but in leap years we have a February 29 and therefore must add one day. Hence the answer is 60 days.

Table of Days for Computing Interest

From Any Day of	To the Same Day of											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
January	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
August	153	184	212	243	273	304	334	365	31	61	92	122
September	122	153	181	212	242	273	303	334	365	30	61	91
October	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	212	242	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

All years whose number is divisible by 4 are leap years and have a February 29, except century years like 1900. Century years are leap years only when their number is divisible by 400. There will be a February 29 in the year 2000. There was none in 1900.

6. General Method of Computing Interest.—Methods of computing interest differ somewhat in different localities. A commonly used method is as follows:

Find the interest on $360 from January 6, 1920, to September 3, 1922, at 6% per annum. The time is 2 years and 240 days. Counting 360 days to the year equals $2\frac{2}{3}$ years. Interest on $360 for $2\frac{2}{3}$ years at 6% is therefore $\$360 \times .06 \times \frac{8}{3} = \$57.60$.

Another way is by what is known as the *Six Per Cent Method.* At 6% the interest for one month is $\frac{1}{2}$ cent on $1. For one day the interest is $\frac{1}{6}$ of a mill $= \$0.000\frac{1}{6}$. The rule generally employed is to call one-half of the months cents, one sixth of the days mills and multiply this sum by the number of dollars.

As worked out this Six Per Cent Method would be as follows:

$\$0.12$ = the interest on $1 for 2 yrs. at 6%.
0.04 = the interest on $1 for 240 da. at 6%.
$\$0.16$ = the interest on $1 for 2 yr. 240 da. at 6%.
360
$\$57.60$ = the required interest.

If we reckon the time in years, months and days, the period covered is 2 years, 7 months, 28 days.

$0.12 = the interest on $1 for 2 yr. at 6%
0.035 = the interest on $1 for 7 mo. at 6%
0.0047 = the interest on $1 for 28 da. at 6%
$0.1597 = the interest on $1 for the required time at 6%.
360
95820
4791
$57.492Ø a difference of 11 cents from the former method.

The interest on $1 for 2 months at 6% is one cent, hence it is obvious that the interest on $2356 for the same time and at the same rate will be $23.56. All that is necessary is to move the decimal point two places to the left.

7. Exact Interest.—Exact interest is used by the United States Government, by some state governments and by many trust companies. The time in our interest problem, when exact interest is to be computed, is $2\frac{240}{365} = 2\frac{48}{73}$ years. The answer is obtained as follows:

Interest for 1 yr. = 360 × .06 = $21.60; for 2 yrs. = $43.20
$\frac{48}{73}$ of $21.60 = 14.20
Total exact interest = $57.40
Principal = 360.00
Amount = $418.40

Suppose we ask ourselves what principal will amount to $672 in two years at 6%? Since $1 at 6% will amount to $1.12 in 2 years, it follows that $672 ÷ 1.12 will give us the principal which in two years will amount to $672. This amount is therefore $600.

8. Compound Interest.—If interest, instead of being paid, is added to the principal at the end of each period as it becomes due, to form a new principal for the next period, the entire interest—that is, the differ-

ence between the final amount and the original principal —is the *compound interest* of the original principal.

A person who placed $1000 at 4% simple interest would of course receive $40 a year. In five years the interest his $1000 capital had earned for him would therefore be $200. But if he loaned his money at 4% compound interest for five years, his interest at the end of the term would be $216.65 or $16.65 more than in the case of simple interest.

This simple problem is illustrated as follows:

Principal	$1000.
Interest for 1st yr. at 4%	40.
Principal for 2nd yr	1040.
Interest for 2nd yr. at 4%	41.60
Principal for 3rd yr	1081.60
Interest for 3rd yr. at 4%	43.264
Principal for 4th yr	1124.864
Interest for 4th yr. at 4%	44.995
Principal for 5th yr	1169.869
Interest for 5th yr. at 4%	46.794
Amount	1216.653
Original Principal	1000.00
Compound Interest	$ 216.65

Had the Indians put the $24.00—the money value of that which Peter Minuit, the first governor of New Netherlands, paid them for Manhattan Island in 1626 —at 7 per cent compounded yearly, the amount would today be about $8,000,000,000. This is more than the assessed valuation of Manhattan at the beginning of the war. The Indians might accordingly have bought back the island with all its improvements and re-occupied New York at almost any time had they had the proper banking facilities.

INTEREST

Compound Interest Table

Showing the amount of $1 at various rates, coumpound interest from 1 to 20 yrs.

Yrs.	2½%	3%	3½%	4%	5%	6%
1.	1.02500	1.03000	1.03500	1.04000	1.05000	1.06000
2.	1.05062	1.06090	1.07122	1.08160	1.10250	1.12360
3.	1.07689	1.09272	1.10871	1.12486	1.15762	1.19101
4.	1.10381	1.12550	1.14752	1.16985	1.21550	1.26247
5.	1.13140	1.15927	1.18768	1.21665	1.27628	1.33822
6.	1.15969	1.19405	1.22925	1.26531	1.34009	1.41851
7.	1.18868	1.22987	1.27227	1.31593	1.40710	1.50363
8.	1.21840	1.26677	1.31680	1.36856	1.47745	1.59384
9.	1.24886	1.30477	1.36289	1.42331	1.55132	1.68947
10.	1.28008	1.34391	1.41059	1.48024	1.62889	1.79084
11.	1.31208	1.38423	1.45997	1.53945	1.71033	1.89829
12.	1.34488	1.42576	1.51106	1.60103	1.79585	2.01219
13.	1.37851	1.46853	1.56395	1.66507	1.88564	2.13292
14.	1.41297	1.51259	1.61869	1.73167	1.97993	2.26090
15.	1.44829	1.55796	1.67534	1.80094	2.07892	2.39655
16.	1.48450	1.60470	1.73398	1.87298	2.18287	2.54035
17.	1.52161	1.65284	1.79467	1.94790	2.29201	2.69277
18.	1.55965	1.70243	1.85748	2.02581	2.40661	2.85433
19.	1.59865	1.75350	1.92250	2.10684	2.52695	3.02560
20.	1.63861	1.80611	1.98978	2.19112	2.65329	3.20713

Yrs.	7%	8%	9%	10%	11%	12%
1	1.07000	1.08000	1.09000	1.10000	1.11000	1.12000
2.	1.14490	1.16640	1.18810	1.21000	1.23210	1.25440
3.	1.22504	1.25971	1.29502	1.33100	1.36763	1.40490
4.	1.31079	1.36048	1.41158	1.46410	1.51807	1.57351
5.	1.40255	1.46932	1.53862	1.61051	1.68505	1.76234
6.	1.50073	1.58687	1.67710	1.77156	1.87041	1.97382
7.	1.60578	1.71382	1.82803	1.94871	2.07616	2.21068
8.	1.71818	1.85093	1.99256	2.14358	2.30453	2.47596
9.	1.83845	1.99900	2.17189	2.35794	2.55803	2.77307
10.	1.96715	2.15892	2.36736	2.59374	2.83942	3.10584
11.	2.10485	2.33163	2.58042	2.85311	3.15175	3.47854
12.	2.25219	2.51817	2.81266	3.13842	3.49845	3.89597
13.	2.40984	2.71962	3.06580	3.45227	3.88327	4.36349
14.	2.57853	2.93719	3.34172	3.79749	4.31044	4.88711
15.	2.75903	3.17216	3.64248	4.17724	4.78458	5.47356
16.	2.95216	3.42594	3.97030	4.59497	5.31089	6.13039
17.	3.15881	3.70001	4.32763	5.05447	5.89509	6.86604
18.	3.37993	3.99601	4.71712	5.55991	6.54355	7.68996
19.	3.61652	4.31570	5.14166	6.11590	7.26334	8.61276
20.	3.86968	4.66095	5.60441	6.72750	8.06230	9.64629

REVIEW.

1. Illustrate the two methods of finding the rate per cent when the base and percentage are given.

2. In reckoning discount, what is the rule for converting two successive rates of discount into an equivalent single rate?

3. What do profit and loss equal in a deal without expenses?

4. What is the difference between a commission merchant and a broker? How is commission or brokerage reckoned?

5. How do you reckon time for which simple interest is paid? State the difference between common interest and exact interest.

6. What is compound interest?

CHAPTER IV

PLANE GEOMETRY—LINES AND ANGLES

1. Properties of Space.—In arithmetic we were interested chiefly in quantities or numbers, which we considered as made up of elements that could be registered or counted and then compared with quantities similarly enumerated. Geometry, however, treats of comparisons of a broader type than those of arithmetic, many of its quantities not being countable at all.

The science of geometry originated in Ancient Egypt where the river Nile, in its tremendous overflows, repeatedly destroyed all land-marks and means of identification over wide expanses of fertile farming country. New and richer land had to be apportioned to the tiller of the soil, to recompense him for the loss of his former fields. It was not possible to retain original position and shape, but it was necessary that the new farm be the same size as the old one. And so, in course of time, four properties of space came into recognition—properties which now distinguish the Science of Geometry. They are *size*, *shape*, *position* and *motion*. A basket-ball, a base-ball and a grain of shot have the same shape, but not the same size. No two of them could lie in the same place at the same time. If we can imagine the space occupied by one of these bodies apart from the material object itself, we will have a *geometric solid*. We might almost call a geometric solid the ghost body of a material solid. A geometric solid is defined as a portion of space, yet space itself is a fundamental conception which it is impossible to limit, define or even fully describe. Our geometric solids have these four

independent properties of size, shape, position and motion. Size can be changed without affecting shape, position or motion, and two of these geometric solids (being "ghosts") could occupy the same position and have the same shape, size and motion at the same time.

2. Dimensions.—When primitive man looked at himself he observed, at first uncomprehendingly, that he had what we call length, breadth and thickness. He saw these same three dimensions in animals, tho he observed that they varied greatly. When he looked at the sky, his eyes saw the heavens to be apparently a sphere no matter in what direction he turned his gaze. It is probable that such simple observations were the origin of the conception that we live in a world of three dimensions—*length, breadth and thickness.*

Our Indians as well as the early Romans thought that a field twice as far around as another was twice as large. A certain lady wanted a box for storing winter clothing. She wanted it to be twice as large as her old one. She directed the carpenter to make it twice as long, twice as high and twice as wide—and on receiving it was greatly surprised at its unexpected capacity. The Indians, Romans and the good lady all thought in one dimension. They did not realize that if we lived in a one-dimensional space everybody could walk a tight rope. There would be no dimension to fall into except along the rope, tho the rope would be narrower than the finest spider-web.

Again, the size of the top of a measure will not tell us the capacity of the measure. We need depth as well. There would be no danger of a skater's falling on the ice in a two dimensional world. He could not fall down.

3. Geometric Elements.—Let us look at this subject from another point of view, taking a stone as a material solid to fix our idea of a geometric solid. Primitive man could have rubbed two stones together as is done today with diamonds. By such grinding he would obtain a *plane surface*. This surface, let us remember, is in space, but is not a part of the solid. It is a part of the *boundary* of the solid, but having no thickness it has one dimension less than the solid, tho still retaining both position and movability. A thin piece of paper will help us to visualize a surface. If, now, the same stones were again ground together until there was obtained another plane surface which met the first, their junction would form a straight line. This straight line, remember, is in space. It is no part of the plane surfaces, but is their boundary. It has position and can be moved. The crease of a piece of folded paper will help to visualize a straight line.

Suppose that still more plane surfaces were formed, some would meet in other straight lines; where straight lines met a *point* would be located. This point, like any straight line, is in space. It is not in the line, but is the boundary of the parts of the line. Without shape and size, it still has position and can be moved. By carefully examining a diamond one may get a good idea of these planes, lines and points.

We can conceive a point, having position and ability to move, generating a line. *A line is the path of a point.* It has one dimension, length. If this line moves in any way except along its own dimension it will generate a surface, for *surface is the path of a line.* A surface has two dimensions, length and breadth, and if this surface moves in any way except in its own two dimensions, it will generate a solid. *A solid is*

the path of a surface and has three dimensions. We can obtain nothing more than solids, unless we are able to move in more than three dimensions.

It is evident from the manner in which they were found, that the points of a line, the lines of a surface and the surfaces of a solid are all uncountable. Also there must be at least one point *between* every two distinct points. These points must be in a definite *order*, and their motion from one position to another along the line must be thru *all intermediate* positions.

When planes do not meet, they are called *parallel;* the floor and ceiling of a room are an example. Also when lines which are boundries of the same plane surface, such as the edges of a floor-board, do not meet, they are parallel. When lines that are not boundaries of the same plane surface do not meet, they are called *skew*.

Any position of a point which generates a line could be thought of as the end point of the line already generated, or the beginning point of the new part. Two fixed points limit a line and the stroke between them in length and position. Two nails will fix a lath. Three points, not all of which are in the same straight line, will fix a plane. Every plane must have three such points. Thus a stool will stand on three legs. Four points, not more than three of which are in the same plane, and not more than two in the same straight line, will distinguish a solid.

4. Formal Geometry.—About 365 years B. C., one Euclid became the first professor of mathematics at the first great university—that of Alexandria in Egypt. He collected all the problems that had arisen as a by-product of the attempts of the Greeks to solve

three "impossible" problems. These three problems were (1) the duplication of a cube, that is, the making of a cube twice as big as a given cube; (2) the trisection of an angle, that is, the division of any angle into three equal parts and (3) the squaring of a circle, that is, finding the exact area of a circle in terms of the radius.

Such of these discussions as were about figures and could be constructed with the straight edge and the compass, Euclid put into a great book made up of 13 smaller books, which he called the Elements. He based his arguments on a few stated assumptions and definitions and on this ground-work by a course of *logical* reasoning reared a structure of *theorems*. This book was not intended for persons of immature mind. In fact, Euclid said there was "no royal road to geometry". None but the best minds of the Greeks, however, mastered Euclid, which was abbreviated more and more until the Romans repeated only a few of the theorems. What is commonly taught in schools today is only a small portion of a few of the 13 books.

5. The Geometrician's Tools.—We have more tools than Euclid and we shall approach the subject rather in the observational manner of his predecessors. We should provide ourselves with a *straight edge*, marked in inches and divided into sixteenths, and a *scale* giving millimeters. These will be used in drawing straight lines, making them of required length and measuring other straight lines. A *pair of compasses* is necessary, one leg furnished with a pencil for the purpose of describing circles and, with the help of the ruler, for laying off required distances and measuring distances already laid off.

We should also have a *protractor* to be used in constructing angles for any given number of degrees, for measuring the number of degrees in any given angle and also for determining whether one angle is greater than another.

For the more accurate construction of geometrical figures, the following instruments are also desirable: a pair of *dividers*, with both legs terminating in fine points, these being more accurate than the compasses for measuring and transferring distances; a *T-square*, for drawing horizontals and perpendiculars; a *bevel*, for constructing angles; *parallel rulers*, for drawing parallel lines; a 30° 60° *right triangle* and a 45° *right triangle*. A pair of either of these triangles may take the place of the parallel rulers and also of the T-square. A hard pencil, preferably HHH, pointed like a chisel, will be serviceable.

With regard to the method of drawing, points and ends of lines should be pricked with the sharp points of the dividers. The paper used should be rather smooth and fastened to a drawing-board with thumb tacks. The reader should construct the figures on a larger scale than those in the Text.

6. Elementary Figures—Angles.—A straight line, a broken line, a curved line and an angle are illustrated in Fig. 2. The *size* of the angle does not depend on the length of the bounding lines *AB* and *AC*, but on the opening between them, that is, on the difference of direction of these lines. An angle is usually indicated by using one letter at the vertex—where the lines join, as at *A*—or as shown by three letters *CAB*. Notice that in the latter case the angle is named in the opposite way from that in which the hands of a clock move. If the point about which the lines revolved were not also the end of the lines, a vertical angle A_2 would be formed. Evidently these two vertical angles are equal.

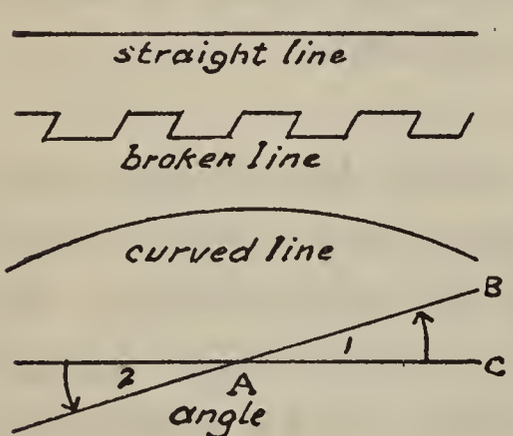

Fig 2.

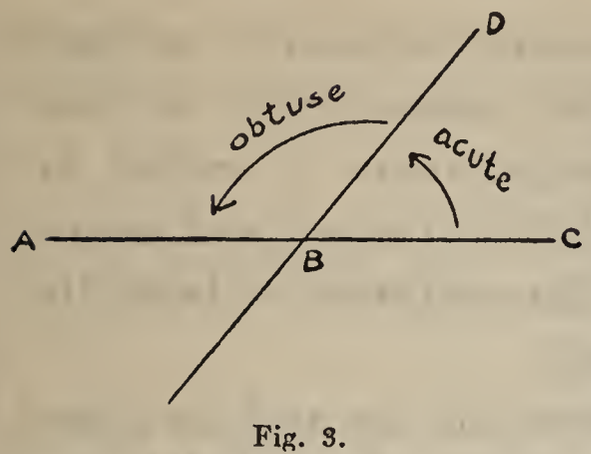

Fig. 3.

If two angles such as *CBD* and *DBA*, Fig. 3, have a common side *DB* between them, and if their other sides are in the same straight line *AC*, they are called *supplementary adjacent angles*. If their other sides are not in the same straight line, they are merely *adjacent*. The angle *EBA*, Fig. 4, is the sum of *EBD* and *DBA*. If the two supplementary adjacent angles *CBD* and *DBA* are equal, each of them is said to be a *right angle* and the lines in the figure are said to be perpendicular to each other. Evidently about *B* in Figs. 3 and 4 there are four right angles. An angle less than a right angle is called an *acute* angle. An angle greater than a right angle is called an *obtuse* angle. Two angles *CBE* and *EBD*, Fig. 4, whose sum is a right angle are called *complementary* angles.

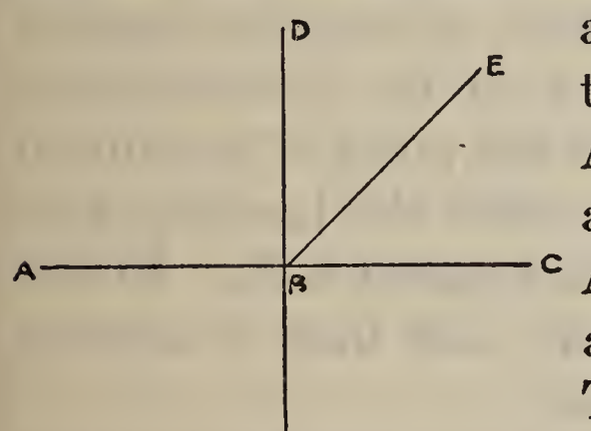

Fig. 4.

7. Elementary Figures—the Circle.—A circle is the usual closed figure described by the compass on a flat surface. Note in Fig. 5, the *center O*, the *radius*, and the *circumference*, also that all radii of the circle are equal, since the compass legs always remain the same distance apart while the circle is being described. When two radii are in line they

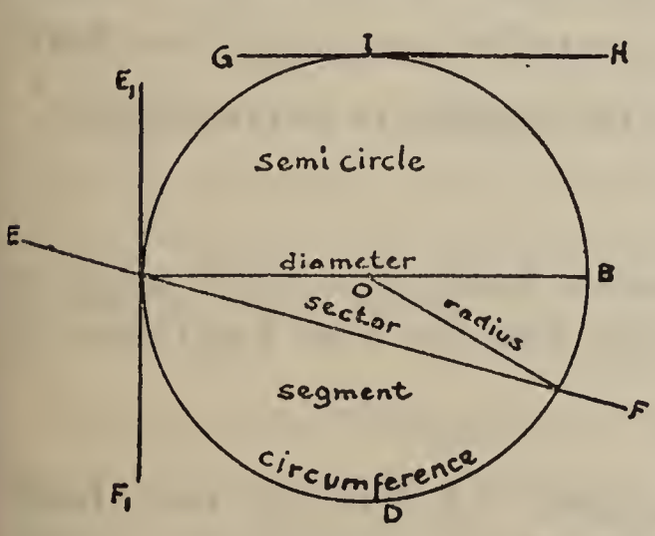

Fig. 5.

form a *diameter*. The portions of the circle on each side of the diameter are called *semi-circles*, or half circles. Any part of the circumference is called an *arc*, as *DB*. A straight line joining the ends of an arc is called a *chord*. If such a line extends without the circle, as *EF*, it is called a *secant*.

The portion of a circle between an arc and its chord is called a *segment*. The part that is bounded by an arc and two radii to its extremities is a *sector*. (See Fig. 5). A semi-circle is both a segment and a sector. A *tangent GIH* is described as a line that cuts a circle in two coincident points—that is, two points in the same place. To illustrate: If the secant *EF* were made to revolve about the point where it cuts the tangent E_1F_1, the point which remains on the circumference would get nearer and nearer to the point of revolution until they became coincident, when the limiting position of the secant would be the tangent E_1F_1. In less scientific language it would be said that a tangent touches a circle at a single point.

Since the time of the Babylonians, at least two thousand years B. C., man has divided the circumference of a circle into 360 arc degrees, written 360°. If an arc contains thirty degrees, we say there are 30° in the angle formed by the arc and by the two radii. Evidently, then, a semi-circumference contains 180° and a right-angle 90°. A 60°-angle is accordingly $\frac{1}{6}$ of 360°.

Thru what angle does the minute hand of a watch move in twenty minutes? Thru what angle does the hour hand move in twenty minutes?

8. Rigid and Non-Rigid Figures.— A triangle has three sides as well as three angles. If we nail three laths

together with three nails we have a rigid figure. We can't change its shape without altering its sides, which would also change its angles. The triangle is therefore fixed if its three sides are fixed. It is also fixed whenever two sides and their included angle are fixed or whenever two angles and their included side are fixed. We can't preserve its size unless we preserve at least one side. Let us ask ourselves whether two triangles are equal if three corresponding parts are the same in each, providing one of these parts is a side? How is this stated for right triangles? Can we preserve a triangle's shape if we preserve only its angles? Obviously an equal triangle can be considered as another position of the first triangle.

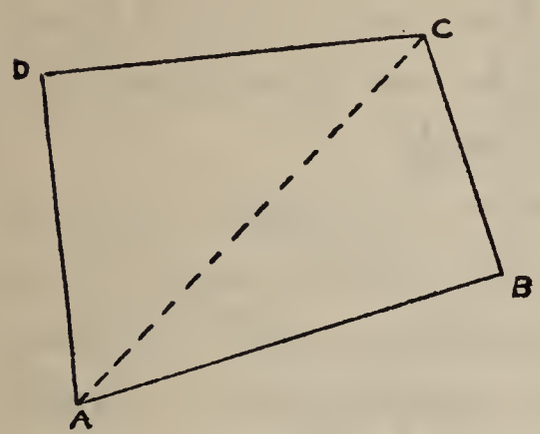

Fig. 6.

A *quadrangle* has four angles. Having four sides, it is also called a *quadrilateral*, which means four-sided. Unlike the triangle the quadrangle is not rigid. Pivots at A, B, C, and D, as in Fig. 6, would allow it to be changed very materially in shape. But if a diagonal AC were drawn between two opposite vertices, the figure would become rigid, as it would then be composed of two triangles.

Observe that an iron bridge is composed of triangles; that is because the triangle is the only rigid *polygon*. (The latter is a common name for closed figures with three or more sides.)

9. Sum of the Angles of Triangles.—Let us draw a line two inches long. Setting our compass points two inches apart with the ends of the line as centers, we

describe short arcs intersecting as at A in Fig. 7. Connecting the point where the arcs cross with the two ends of our straight line we will find that we have made an *equilateral* or equal-sided *triangle*.

Fig. 7.

Next, measure its angles with a protractor. How many degrees are there in each and in all? Is the triangle also equiangular? Could we pave a floor with tiles of this shape? If constructed accurately, such a triangle would have exactly 60° in each angle.

It matters not what shape a triangle has, the sum of its angle always equals 180°. Accordingly, if we turn CB, Fig. 8, thru the angle at C until it falls on CA and then turn it thru the angle at A until it falls along AB, it is in the same line as if it had been turned first to the direction *BE parallel to AC* and then made to fall upon AB extended. We can see that not only is the sum of the angles A and C equal to the angle B_{2-3}, that is, two interior angles of a triangle always equal the opposite exterior angle, but since B_1 and B_{2-3} are together equal to a turn thru 180°, then $A+B+C$ must also be equal to a turn thru 180°.

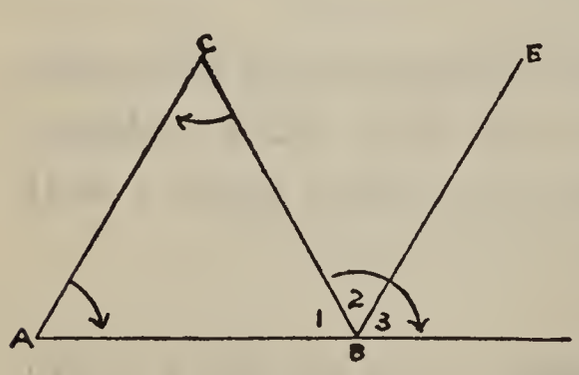

Fig. 8.

If two angles of a triangle are known, so is the third. The angle C equals B_2, Fig. 8. They are called *alternate interior angles of parallel lines.* Also A and B_3 are equal and are called *corresponding angles of parallel lines.*

10. Sum of the Angles of Other Figures.—A *quad-*

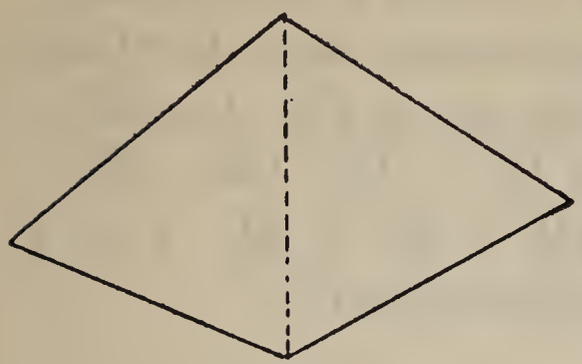

Fig 9.

rilateral, Fig. 9, may be regarded as composed of two triangles. The sum of its interior angles is, therefore, four right angles. The five-sided polygon called the *pentagon*, Fig. 10, may be divided into three triangles—just two less than the number of its sides. Its interior angles are together equal to three times two right angles, that is, they equal six right angles or 540°. If the five angles were all equal, each would accordingly be 108°.

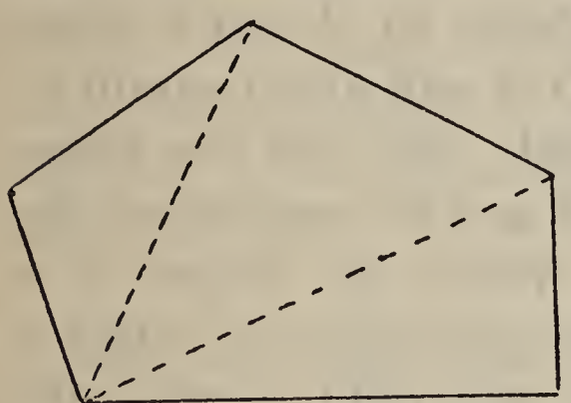

Fig. 10.

It is helpful to remember that we can always determine the number of degrees in the interior angles of any polygon by multiplying 180° by two less than the number of its sides.

Next convince yourself, by measuring the size and angles of triangles by dividers and protractors, that wherever two sides of a triangle are equal (such triangles are called *isosceles*) the opposite angles are also equal, and, conversely, where two angles are equal, the opposite sides are equal.

Is this true also of equilateral triangles? Obviously an equal triangle can be considered also as another position of the same triangle. We would do well to give these facts some consideration and also to assure ourselves that where the sides of a triangle are unequal, the greater angle always lies opposite to the greater side. Thus the *hypotenuse AC*, Fig. 11, being opposite the right angle, is the longest side in the triangle.

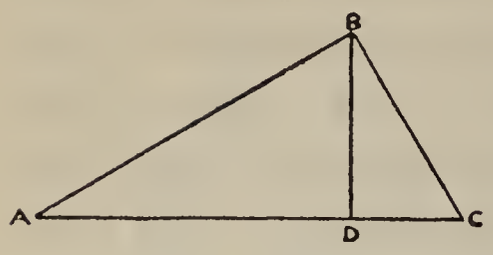

Fig. 11.

Any two sides of a triangle are together greater than the third side, since, as in Fig. 11, $AB > AD$ and $BC > DC$, then $AB + BC > AD + DC = AC$ ($>$ is read "greater than".)

11. The Parallel Rulers.—Not only are the lines AB and DC of our parallel rulers parallel, as in Fig. 12, but by making the points A, B, C, D for any setting, we shall find that if one ruler is placed on A and B when the other contains the latter D it will also contain C. AD and BC will also be parallel. We call the figure $A\,B\,C\,D$ a *parallelogram*. If AB and BC were equal, the figure would be a *rhombus*. When the angles of a parallelogram are right angles the figure is called a *rectangle*, but when they are not right angles, the figure is a *rhomboid*. When the angles of a rhombus are right angles, it is called a *square*. A square is therefore also an *equilateral rectangle*.

Fig. 12.

Note that if the distances AB and BC, Fig. 12, are equal, the angles DAB and DCB will also be equal. Likewise CAB and DAC and ACB and DCA, Fig. 13, are all equal. The angles A and C are bisected and the parallelogram divided into two equal triangles, no matter what position the rulers take. Try to prove this fact in other ways. Also notice that since the sum of the angles CAB and DAC is equal to the angle at B, the exterior angle of the triangle ABC, CAB is one-half of angle at B.

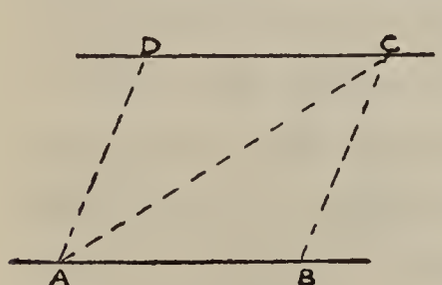

Fig. 13.

Now turn to Fig. 14. The circle has B as the center and BC as a radius. It passes thru A and the sides of the angle A_1 are chords. On account of these facts, A_1 is called an *inscribed* angle. As angle B_2 has as many degrees as the arc CF, angle A_1 will have only half as many. Since C_2 and A_1 are equal, arc EA and arc FC are also equal, having twice as many degrees as angle A_1 or its equal C_2. Hence parallels cut off equal arcs on a circumference. This may be tested by measurement.

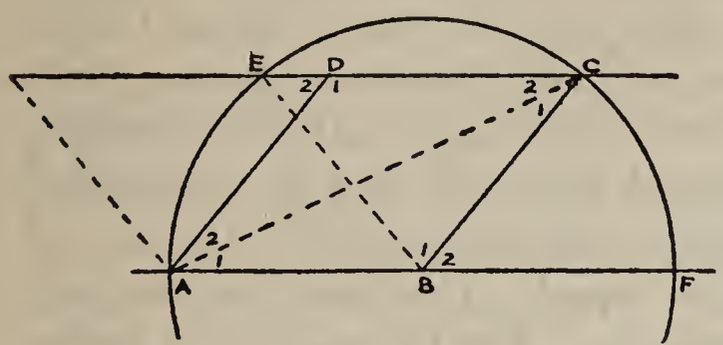

Fig. 14.

Ships avoid rocks and shoals near the coast by what is called in navigation *the horizontal danger angle.* At A and B, Fig. 15, are two light-houses on shore and C is represented on the charts as a dangerous rock with a circle about it containing all the dangerous shoals. The large circle is drawn by trial thru A and B tangent to the small circle, and the angle ADB is measured by the protractor. With an instrument called a sextant, the navigation officer measures the angle APB when P is his own position. As long as APB is less than ADP the ship is outside the large circle and therefore outside the smaller circle where the danger lies.

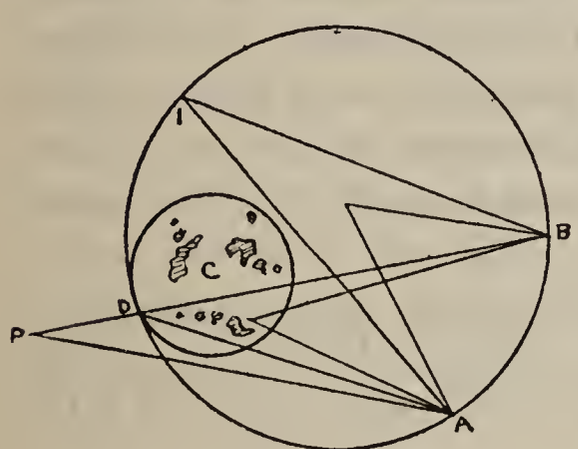

Fig. 15.

12. Bisectors and Perpendiculars.—These tools (Fig. 12) are special parallel rulers—a form of *linkage.* With them we could bisect any angle very neatly, more so in fact than by the protractor or by folding a paper

on which the angle is drawn. The following is an example of this linkage, but does not give parallels.

Let us mark off, with a compass or dividers, equal lengths *AD* and *AE*, as in Fig. 16, also, from centers *D* and *E* describe arcs with equal radii, these arcs meeting at *F*. Then *AF* is the *bisector* of the angle *A*, for the triangles *ADF* and *AEF* are equal, as can be seen by noting their symmetry and tested by folding along *AF*, or by observing that their sides are equal. They are thus two positions of the same triangle. For the purpose of bisecting the angle *A* the lines *DF* and *EF* need not be drawn.

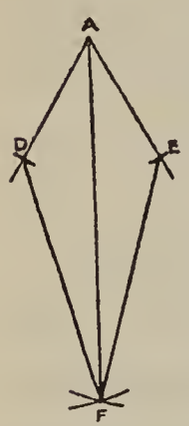

Fig. 16.

The longer the lines and the more nearly at right angles the arcs cut, the more accurate is the drawing likely to be. We can use this same figure, called the *kite*, to bisect a line or an arc. For if we are given either the line *AB*, Fig. 17, or the arc *AB* with center at *D*, then with *B* and *A* as centers and radii equal to *AD* describe arcs intersecting at *C* and *D*. A continuation of the arcs, as drawn in Section 9 of this chapter, would have produced two equal triangles, base to base, that is, isosceles triangles, for since the sides of these triangles are equal, each to each, the three sides of the two triangles *ACB* and *ADB* are equal. Likewise the two triangles *ACD* and *BCD* are equal, having two sides and the included angle equal. *AD* and *DB* are equal, also the two arcs *AF* and *FB*, since the two angles *ADC* and *BDC* are equal.

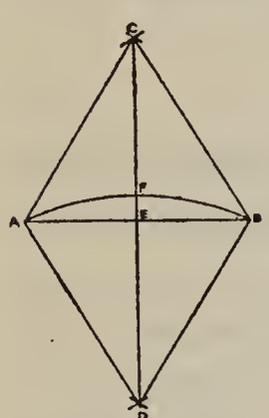

Fig. 17.

Would all this have been true if merely *AC* were equal to *CB* and *AD* equal to *BD?* Try this construction in several forms, measuring carefully, and notice that in all forms of the "kite" *CD* and *AB* are perpendicular and that the perpendicular bisector of a chord also bisects its arc.

13. Drawing Tangent From External Points.—The "kite" may also be used for drawing a tangent from an external point as *P* in Fig. 18, to a circle whose center is *O*. In this case *OP* is bisected at *M* and another circle is constructed with center *M* and radius

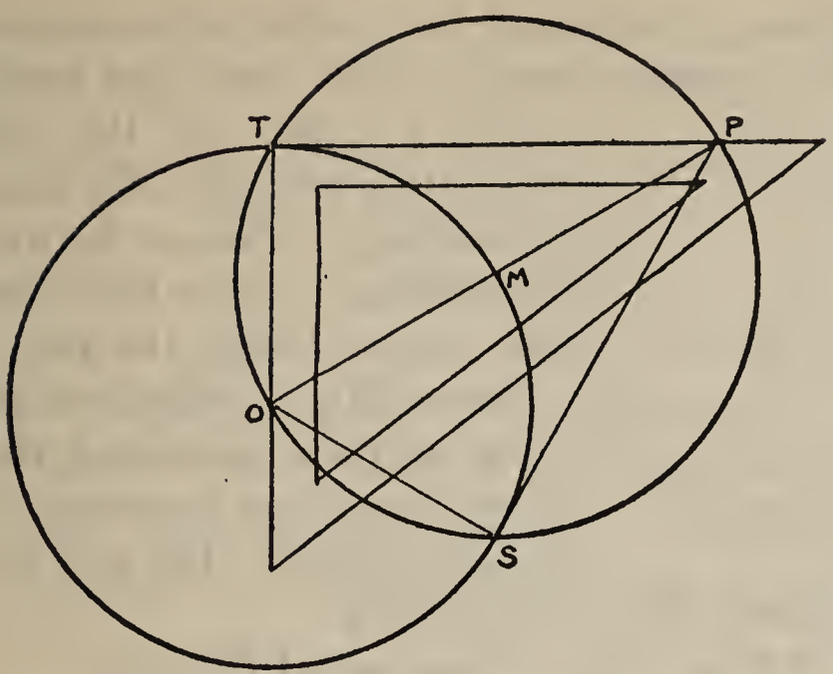

Fig. 18.

MP, meeting the first circle at T and S. It will be found that the angles OTP and OSP are right angles, being measured by half of 180°. The symmetry of the kite also shows that these tangents PT and PS are equal and make equal angles with the line drawn from the point P to the center.

An easy way to draw a tangent to a circle is to use a right triangle with one leg along the radius and the vertex of the right angle on the circumference. If the point is on the circle, the right angle is at the point, but if the point is without the circle, the triangle is so revolved that the other leg of the triangle just touches the point as in the figure.

14. Points at Which Two Circles Meet.—The circumferences of two unequal circles never meet in more than two points.

Let us cut two circles out of a piece of paper or use two coins, sliding them over each other. If we were to fasten the two paper circles with glue and fold them about their common diameter, so that, as in Fig. 19, the point Q falls on the point P, on unfolding them it will be seen that the common chord is bisected at M. We might say that the figure APBQ is a kite and evidently BMQ and PMB are right angles.

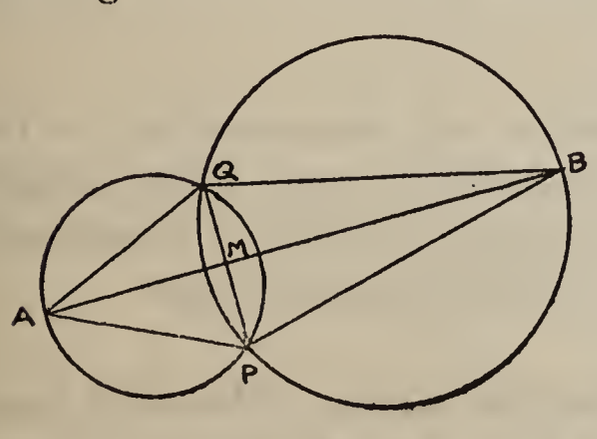

Fig. 19.

We note that the line of centers of two circles is the perpendicular bisector of their common chord. If the circles just touch, as in Fig. 20, will the lines of centers pass thru the point of contact? This can be tried by folding. Notice that when the circles touch, the line of centers OO_1 is equal to the sum of their radii and that they have one common internal tangent, AB, and two common external tangents, $T_1\ T_2$ and T_4T_3.

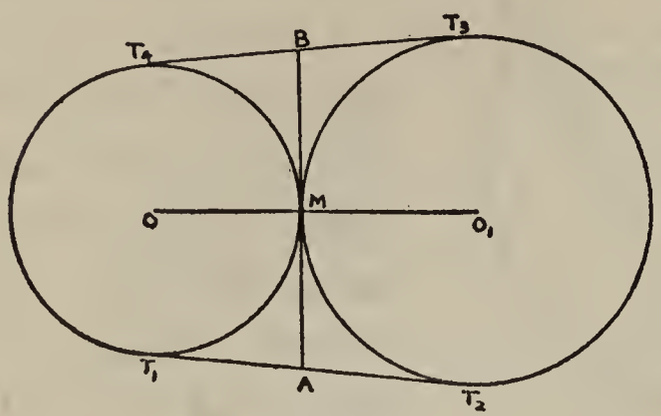

Fig. 20.

Evidently, $T_1A = AM = AT_2$, so they equal BM, etc.

When the two circles just touch internally, there is but one common external tangent and the line of centers equals the difference of the radii. What happens, we may ask, when the line of centers is greater than the sum of the radii? When it is less than the sum but greater than the difference? When it is less than the difference?

15. Perpendicular from Outside a Line.—By referring to Fig. 17 of this chapter, we can draw a perpendicular from a point D outside a line by the following device: With D as the center and with a radius greater than the distance to the line (by *distance* we always mean perpendicular distance) an arc is constructed by cutting the line at A and B. Then with A and B as centers we construct arcs of equal radius meeting at C, thus forming a kite, whose diagonals are necessarily perpendicular.

We could, of course, have drawn any perpendicular to the line and run a parallel to it thru the point by means of the parallel rulers or some other device. In practice, however, it is generally better to use the set triangles in drawing perpendiculars.

16. Bisecting Lines with Parallel Lines.—Parallel lines can be used to bisect lines and in fact to divide them into

any number of equal parts. Suppose we wish to bisect *AB*, Fig. 21. In order to make any convenient angle with *AB* we draw *AC*, then with our dividers mark off *DE* equal to *AD*. Following that, the line *BE* is drawn and then *FD* parallel to *BE* and *FG* parallel to *AC*. The two triangles thus formed, *ADF* and *FGB*, are equal and *F* is the middle point of *AB*. The construction may be extended to a division into three or more parts.

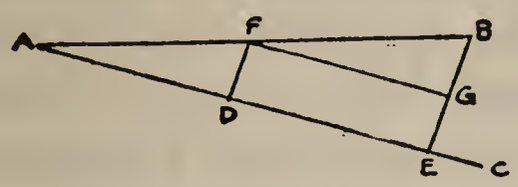

Fig. 21

17. Dividing Line with Rectangular Scale.—We can advantageously use the scale made by dividing either or both the length and breadth of a rectangle into a number of equal parts. Mark the lower left-hand corner of the rectangle *A*, as in Fig. 22, and the points of division in either direction 1,2,3,4, etc.

Let us suppose we want to divide the line *AB* into five equal parts. If our rectangle is transparent, we could place it over the line with *A* at the point *A* and move it about until *B* falls on the perpendicular from 5. Then where the perpendiculars from *1,2,3,4* cross the line we have the points of division. If the rectangle is not transparent, we can take the length of the line *AB* with the dividers, putting one end at *A* and seeing where the other crosses the perpendicular from 5. Hold the dividers in place and put a straight edge against their points. This reproduces the line *AB* and the points are found as before. It is well to have a smaller scale on one side of the rectangle, so that lines of various lengths may be conveniently divided.

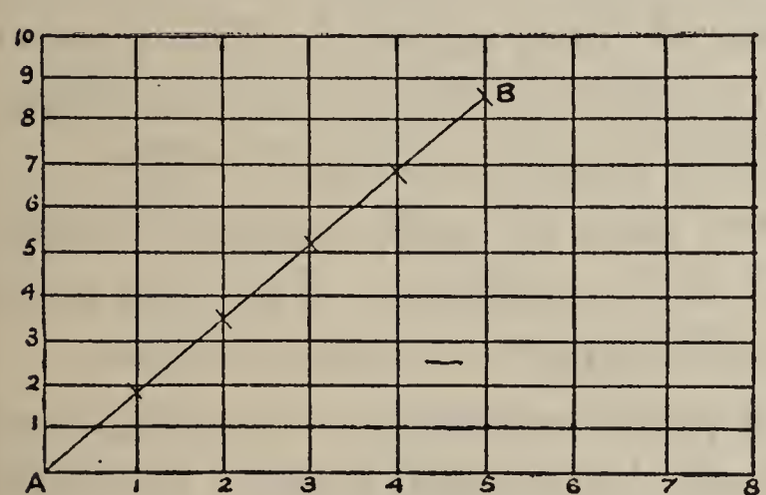

Fig. 22.

18. Concurrent Lines in Triangles.—The problem involving lines that meet in a triangle can be solved as follows: Cut from paper several triangles of different shapes. Take one of them and fold so as to bring two vertices together. The fold will form a perpendicular bisector of the side to which the two vertices belong. Do this carefully for all three sides and notice that the *perpendicular bisectors* meet in a point. Then measure the distance of this point from each vertex of the triangle. These distances should be the same. Next take another triangle and fold one side along itself so as to obtain a perpendicular from the opposite vertex. Do this with each of the other sides and note that the three *altitudes* meet in a point. Now fold still another triangle so as to obtain the *bisectors* of each of the *angles*. These three bisectors will meet in a point. Measure the perpendicualr distance from this point to each of the opposite sides. They should be equal. Continuing the experiment, mark the midpoints of each side of another triangle and make a fold extending from one of these points to the opposite vertex and similarly for all the other points. Notice that these three *medians* meet in a point. Then set dividers to the shortest part of each median in turn and step off the length of the medians. The long parts will be just twice the short parts in each median.

Having reached this point we require an equilateral triangle, which can easily be devised from a carefully made square of paper, first folding it in half as if closing a book, then folding over without creasing one side from the left vertex till it just meets the middle fold. Mark this point on the middle fold and make a crease to each of the most distant corners of the square. If this is done carefully we will have an accurate equi-

lateral triangle. Now cut off the excess parts and attempt to go thru the preceding exercise. If the construction is accurate, it will be found that the bisectors of the angles and sides are the same, as are also the medians and altitudes and that they all meet at the center of the inscribed and circumscribed circles.

By constructing the circumscribed circle and the inscribed circle of an equilateral triangle it will be found that the *radius* of one is twice the radius of the other. It is advisable to study these figures carefully, as they will be referred to again and again.

19. Symmetry.—Symmetry is largely a matter of balanced contrariety. Our two hands are symmetrical with respect to each other, but we can't put our right-hand glove on our left hand. So also are parts of a "kite" on each side of the long diagonal; so also a *rhombus* or a square with respect to either diagonal, and a circle with respect to every diameter. Every line thru the intersections of the diagonals of a rhombus or a square and terminated by the boundary is bisected at the point mentioned. Such a point is called a *center of symmetry*, while the diagonals are called *axes of symmetry*.

REVIEW.

1. What are the properties of space? How many "dimensions" are there? Name them.

2. Define "point," "line," "surface" and "parallel."

3. Explain the use of the tools required by the geometrician.

4. What is an angle? A circle? A segment? A sector? An equilateral triangle? A quadrilateral? The hypotenuse? A parallelogram?

5. What do you know of the "kite?"

CHAPTER V

PLANE GEOMETRY—AREAS AND LOCI

1. Principle of Area.—Equivalent figures are those which have the same size. If a rectangle contains a given unit of length in its length and breadth, it can be divided like the figure into little squares which will be the square unit. In Fig. 23 there are five rows of nine squares each; hence there are 45 square units in the area. So for all rectangles we will take the product of the number of linear units in the length by the number in the breadth and call this the number of square units in the area. This same method is followed when length and breadth do not contain the exact linear unit. The side on which a figure stands is called its *base;* its *altitude* is the extreme height above that base.

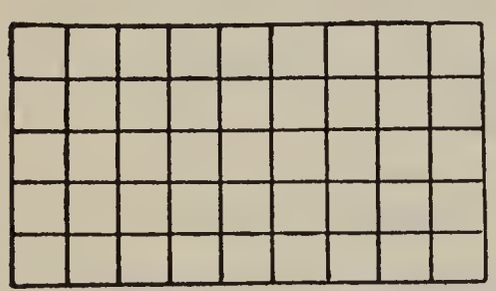

Fig. 23.

2. Area of Parallelogram.—Any *parallelogram* (a four-sided plane figure whose opposite sides are parallel) can be turned into a rectangle by cutting it in two the shorter way, that is, by drawing a line perpendicular to its base, and moving the part over as in Fig. 24. The parallelogram $ABCD$ and the rectangle $A_1B_1C_1D_1$ are equivalent in area and they evidently have the same base and altitude. Without troubling to cut the parallelogram, however, we can say that its area is

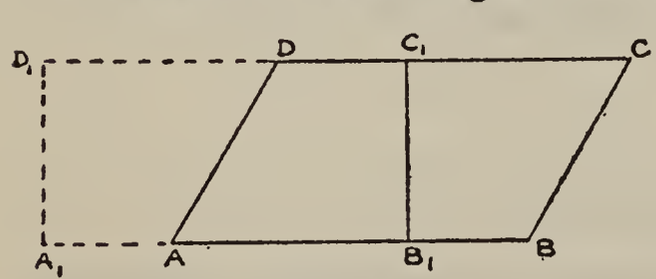

Fig. 24.

found by multiplying the number of units in its base by those in its altitude and calling this the number of square units of area.

3. Area of Triangles.—Any triangle can be revolved about the midpoint of one of its sides as a center of symmetry and when the two halves of that side have been brought into coincidence the whole will form a parallelogram, as in Fig. 25. As the area of a parallelogram was expressed by base times altitude, the area of the triangle which has the same base and altitude as this parallelogram is evidently expressed by $\frac{1}{2}$ base times altitude. This last expression means either one-half the base times the altitude or one-half the altitude times the base or one-half their product. Thus, a triangle whose base is 8 and altitude 6 has an area of $\frac{1}{2}(8 \times 6)$, which equals 4×6 or 8×3 or $\frac{1}{2}(48) = 24$. It is evident that a triangle with an altitude of 8 and a base of 6 is equivalent to a triangle with a base of 4 and altitude 12, and that if the bases of two triangles are the same and the altitude of one is twice the altitude of the other, the former triangle must have twice the area of the latter.

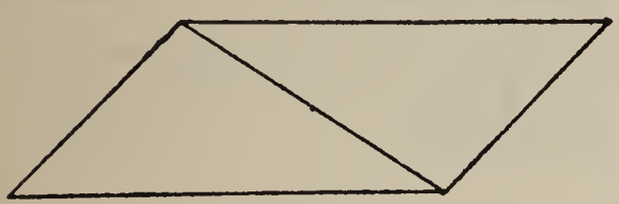

Fig. 25.

4. Area of the Trapezoid.—A *trapezoid* is a quadrilateral, two of whose sides are parallel. In order to get its area we usually divide it into two triangles each one of which has the altitude A, Fig. 26. When one triangle has the base B and the other the base B_1, the area of the trapezoid is evidently

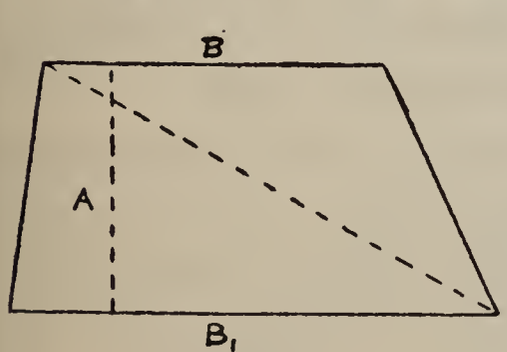

Fig. 26

$$\frac{1}{2}AB + \frac{1}{2}AB_1 = \frac{A}{2}(B + B_1) \text{ or } \frac{A(B+B_1)}{2}.$$

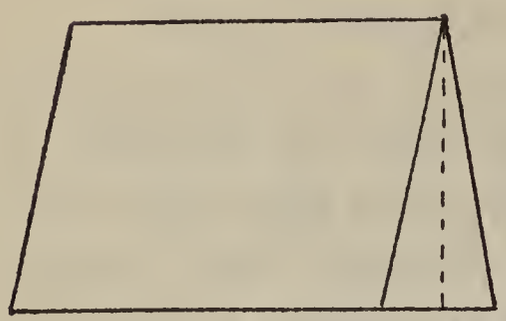

Fig. 27.

But in advanced work the trapezoid is generally divided into a parallelogram and a triangle, as in Fig. 27. Here if B (Fig. 26) were 6, B_1, were 8, and A were 4 we could say the area of Fig. 27 is $6 \times 4 + \frac{1}{2}(4 \times 2) = 28$ instead of $\frac{4(6+8)}{2}$.

5. Area of Trapezium.—The *trapezium* is a quadrilateral none of whose sides are parallel. Like all other polygons, it is turned into triangles by the following method: As triangles have fewer corners than any other figure, it is desirable to cut off a corner of the trapezium *DABC*, Fig. 28, and therefore the diagonal *DB* is drawn. Now draw *CE* parallel to this diagonal, meeting *AB* produced at *E*. Next draw *DE*. As parallels are everywhere equidistant, the altitude of the two triangles *DCB* and *DEB* are equal, and as they have the same bases they are equivalent and we may replace *DCB* by *DEB* and the quadrilateral will have been reduced to the triangle *ADE* of which we could readily find the area.

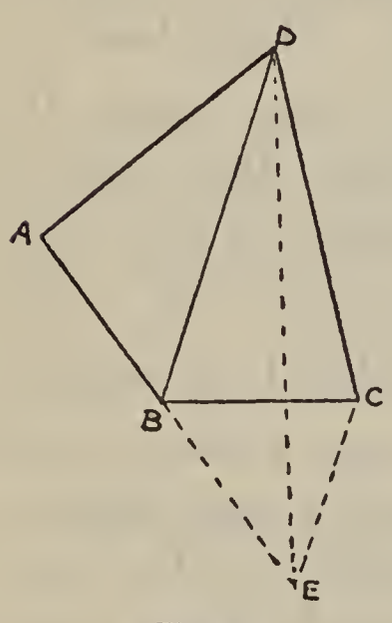

Fig. 28

It would be well for the reader to convince himself by accurate measurement that these things are true, then to keep his data until he comes to trigonometry, when he can compare them with results obtained by new methods learned there. It is admirable practice for the student of mathematics to master a piece of work

and then pigeonhole it in his mind till he meets with the same problem in another setting.

6. Equivalent Parallelograms.—There is an interesting method of constructing a parallelogram with a given base that will be equivalent to another given parallelogram—a method which will be brought into frequent use later on. The given parallelogram is *AFEK*, Fig. 29, and the given base of the new parallelogram is *EG*. First complete the parallelogram *KEGD* and draw the diagonal *DE* to meet *AF* produced to *B*. By drawing *BC* parallel to *AD* and extending *DG* to *C* and *KE* to *H* we have the diagonal of each of the parallelograms *BHEF*, *EGDK*, and *BCDA*, and therefore we have bisected each of them. Now since the triangle *ABD* equals the triangle *CBD*, and *FBE* equals *HBE*, and *KED* equals *GED*, if we subtract the sum of the last two from the first, we see that parallelogram *AFEK* is equal to parallelogram *EGCH*.

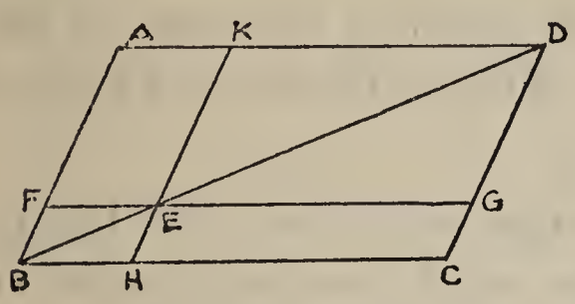

Fig 29.

Considering areas as products of two factors, show by measurements with dividers that in Fig. 30 $FE \times EK = EG \times EH$, where the lines and letters refer to both Figs. 29 and 30. Make other such figures and convince yourself that *if two chords of a circle cut each other within the circle the rectangle contained by the segments of the one is the rectangle contained by the segments of the other.* Is this the same as the following: Of all chords intersecting at a point within a circle, the product of the parts of one chord equals the product of the parts of any other chord? In other words, if one chord had parts of 8 and 2 inches respectively,

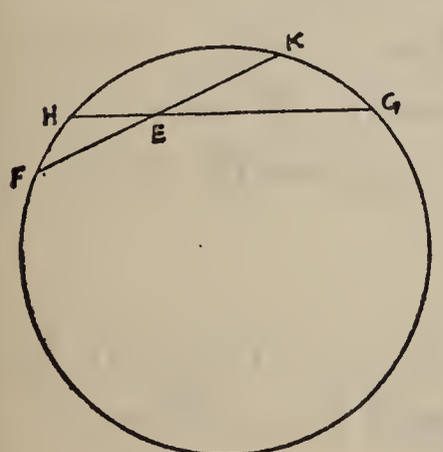

Fig. 30.

would the product of all such segments of other chords also be 16? An attempt to answer this question will strengthen our grip on these problems.

7. The Square of the Hypotenuse.—*The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.* This is *The Great Theorem of Pythagoras.*

The "harpedonaptae" or rope stretchers of Egypt four or five thousand years ago made perpendiculars to given lines by forming right triangles whose sides were as 6, 8, 10 or 3, 4, 5, as the case may be. Here $6^2 + 8^2 = 10^2$; that is, $36 + 64 = 100$. Modern surveyors measure 40 feet, then 80 feet, dividing the latter into 30 and 50. Three men stretch the tape—one at each end, *A* and *B*, Fig. 31, and one at the division point *C*. Their positions form a right triangle with a right angle as marked at *A*. In order not to bend and break the tape, the man at the point of division *C* brings together parts 30 feet from one end and 50 feet from the other end of the 100 foot tape, the whole of which he uses, 20 feet forming a loop at *C*.

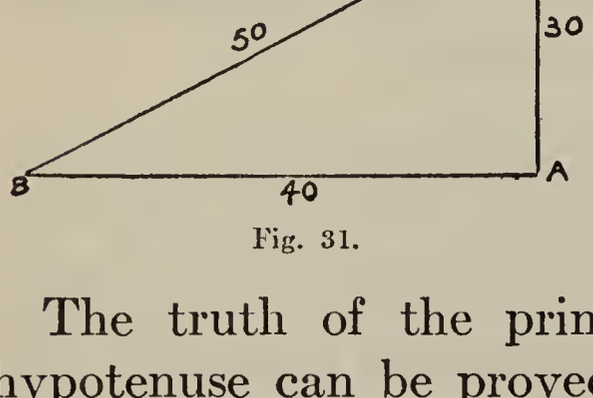

Fig. 31.

The truth of the principle of the square on the hypotenuse can be proved in thousands of ways, but probably less than one hundred are generally known, and only one need be given here.

8. Pythagoras' Proof.—The proof that is thought by some to be that by Pythagoras himself is as follows:

Make two large squares of equal size or area, as (*a*) and (*b*), Fig. 32. Divide each side into a long part and a short part, the

corresponding parts being of the same length, as shown. Then draw the two figures, doing so with care. The triangles, it will be noted, are all equal to each other. In (*a*) the inner square is that on the hypotenuse *AB*; in (*b*) the squares are those on the two sides *CD* and *DE* Taking away the four equal triangles from each figure, it follows that the remaining square on the hypotenuse is equivalent to the sum of the remaining squares on the other two sides. Use this theorem to find the altitude of an equilateral triangle whose side is 10.

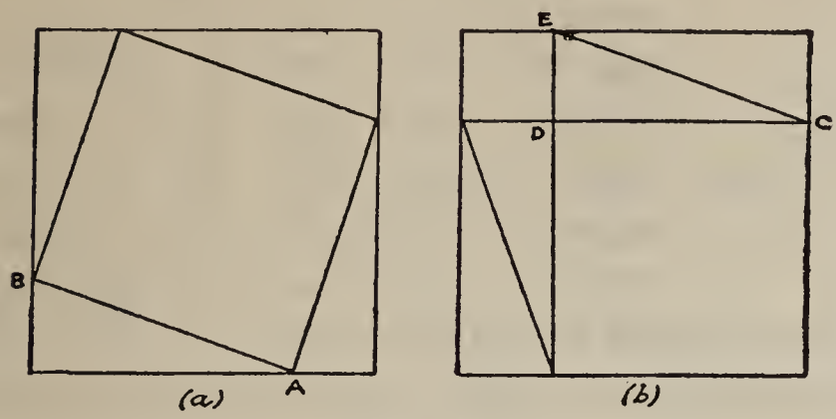

Fig. 32.

9. Square Equivalent to Rectangle.—A square equivalent to the rectangle *ABCD*, Fig. 33 (*a*), may be constructed as follows: Make *DE* equal *DC*, bisect *AE* at *F*, draw a semi-circle with *F* as center and *AE* as diameter, and produce *CD* meeting the circle at *G*. Then the square on *DG* is the square required. Complete this figure and the proof is as in Section 6 of this chapter.

By constructing a rectangle equivalent to a parallelogram this method can be used to construct a square equal to a parallelogram.

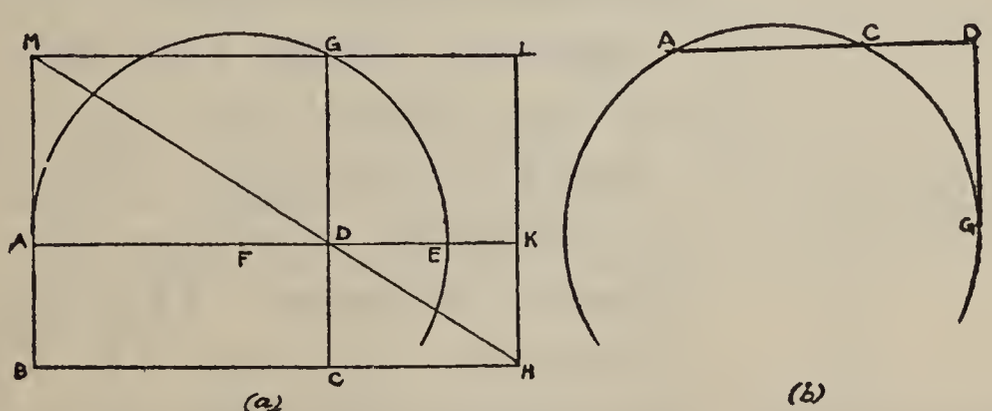

Fig. 33.

By a similar method show that the square on the tangent *GD*,

Fig. 33 (*b*), is equal to the product of the whole secant *AD* and its external segment *CD*. Draw other secants from *D* and convince yourself that the truth is probably general. Also show that in the right triangle *ABC*, Fig. 34, the square on *AB* equals the rectangle contained by *AD* and *AC*; that the square on *BC* equals the rectangle contained by *AC* and *DC*, and that the square on *BD* equals the rectangle contained by *AD* and *DC*.

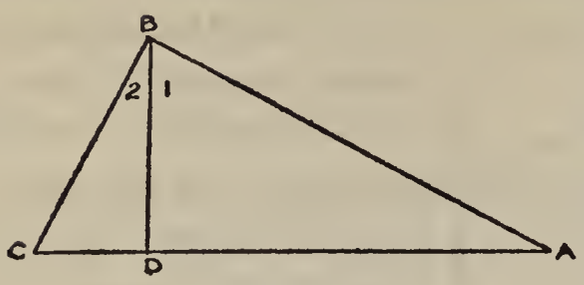

Fig. 34.

The following formula gives a large number of right triangles by substituting for n the values 1, 2, 3, 4, 5:

$$(2n + 1)^2 + (2n^2 + 2n)^2 = (2n^2 + 2n + 1)^2$$

For $n = 1$, this formula becomes $(3)^2 + (4)^2 = (5)^2$

For $n = 2$, this formula becomes $(5)^2 + (12)^2 = (13)^2$

For $n = 3$, this formula becomes $(7)^2 + (24)^2 = (25)^2$

The reader should construct very accurately a number of these right triangles and test them with a protractor and set square.

10. Similar Triangles.—In geometry *similar* means having the same shape. Two triangles are similar when the angles of one triangle are equal to the angles of the other. This is not a sufficient condition for other figures, however. Thus the two rectangles (*a*) and (*b*), Fig. 35, are not similar tho their angles are equal.

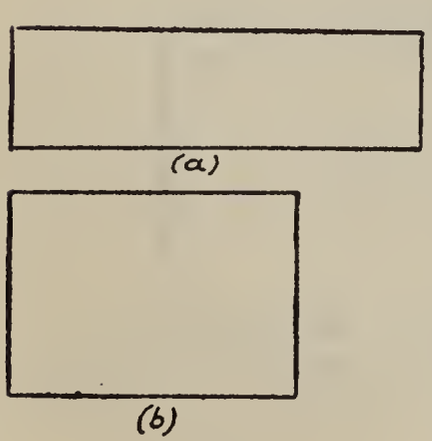

Fig. 35

On a base BC of 15 mm., Fig. 36, construct a triangle with sides AB and AC of 20 and 25 mm. respectively. Then construct another triangle, Fig. 36 with base 30 mm., and make the angle B_1 equal to the angle B and angle C_1 equal to angle C. This will make A_1 equal to A. Hence the two triangles are equiangular and *similar*.

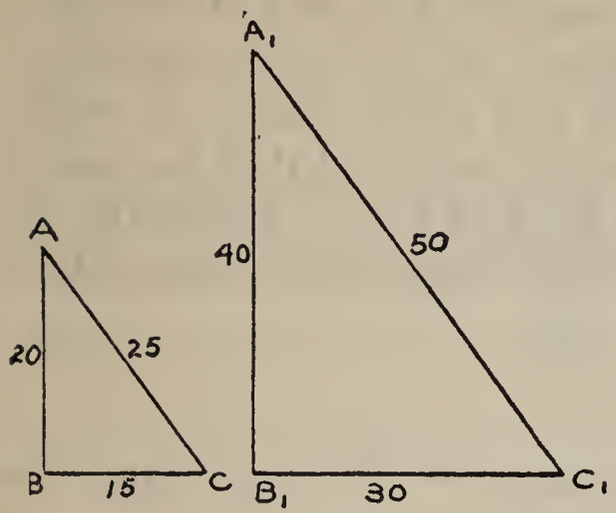

Fig. 36.

Now measure the lengths of the sides of the triangles, which will be found to be as in the figure. Comparing the corresponding sides about the angles in each figure we find the same relation in each case. Thus,

About C, the relation is $\frac{15}{25} = \frac{30}{50} = \frac{3}{5}$

About B, the relation is $\frac{15}{20} = \frac{30}{40} = \frac{3}{4}$

About A, the relation is $\frac{20}{25} = \frac{40}{50} = \frac{4}{5}$

This comparison of the numerical values of the length of two sides is called *ratio*. The statement of equality of two or more ratios is called *proportion*. We have found, in Fig. 36, that the corresponding sides about equal angles are proportional.

The reader may construct for himself other figures taking different numbers for the sides of his first triangle. If we have a triangle whose sides are 3, 5, 7, and a similar triangle with the side corresponding to 3 having a length of 6, we shall find that the other sides are 10 and 14.

11. Demonstration of Principles of Similarity.—The result reached in the two foregoing sections may be demonstrated more generally as follows:

Let the triangles ABC and $A_1B_1C_1$, Fig. 36, be similar triangles and let them be placed so that AB rests on A_1B_1 and AC on A_1C_1. Then BC is parallel to B_1C_1. Suppose AB and A_1B_1 commensurable, and let AB contain n units and A_1B_1 contain n_1 units. Further, suppose A_1B_1 divided into its units and thru the points of division draw lines parallel to BC and B_1C_1. Evidently the divisions of A_1C_1 are all equal, tho not necessarily equal to those of A_1B_1. Hence $\frac{AB}{A_1B_1} = \frac{n}{n_1} = \frac{AC}{A_1C_1}$

In like manner the proportionality of the sides about the other equal angles may be shown.

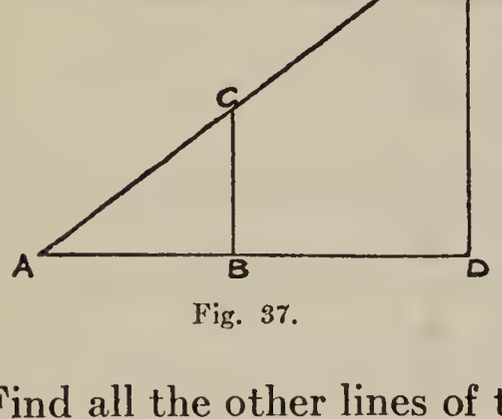

Fig. 37.

In Fig. 37 the triangles ABC and ADE are similar. The lengths of the sides are given, AB being equal to 32; BC to 24 and AD to 75. Then the property of the similar triangles gives

$$\frac{DE}{75} = \frac{24}{32}$$

$$DE = \frac{24}{32} \times 75 = \frac{225}{4} = 56\frac{1}{4}$$

Find all the other lines of the figure.

12. Range Finders.—In the Navy, range finders are often constructed on the following principles: A base line, say 300 feet, is measured along the deck of a vessel over the keel. Telescopes are placed at each end turning on horizontal circles registering 0° each when pointed at each other. When a hostile ship appears in sight these telescopes are trained on some part of it, say a smokestack, and the angles read and telephoned. The computer constructs on a 1-foot base line a similar triangle by making the angles C and A, Fig. 38, equal

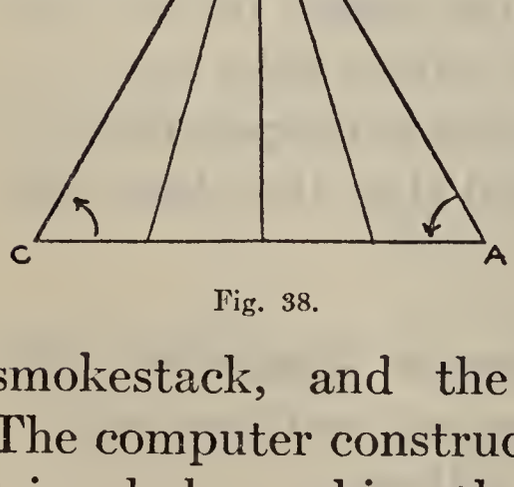

Fig. 38.

to the observed angles. Then every line of the triangle represents a line 300 times its length on the sea triangle, and the range is determined with ease.

13. Enlarging Figures.—The proportionality of the sides of similar triangles may be used to reduce or enlarge any sort of figures to any scale as seen by the diagram. This is very useful in drawing maps and enlarging or reducing plans. An instrument called the pantograph will do the work more quickly, but only those that are expensive are accurate. In Fig. 39 distances in the larger quadrangle *ABCD* are double those in the smaller one *EFGH*.

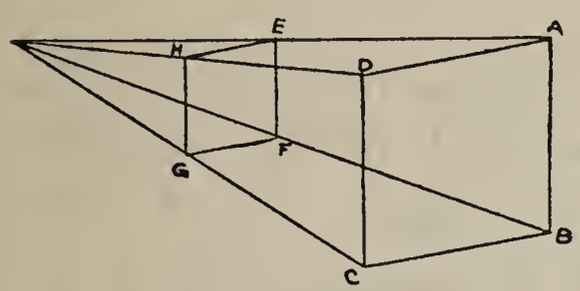

Fig. 39.

14. Comparison of Areas of Similar Triangles.—Let any side, say the base *AB* of a triangle *ABC*, Fig. 40 (*a*), be divided into as many parts as it contains units of length. Thru the points of division draw lines parallel to the sides and thru the points of intersection of these lines draw lines parallel to the base. The triangle is thus divided into a number of equal triangles each one of the same shape as the original triangle. Then draw another similar triangle, as Fig. 40 (*b*). The latter contains $1 + 3 + 5 = 3^2$ small triangles; the former contains $1 + 3 + 5 + 7 = 4^2$ small triangles. As the bases of the two triangles are 3 and 4 and their areas are to each other as 3^2 to 4^2, we are led to the statement that *similar triangles are*

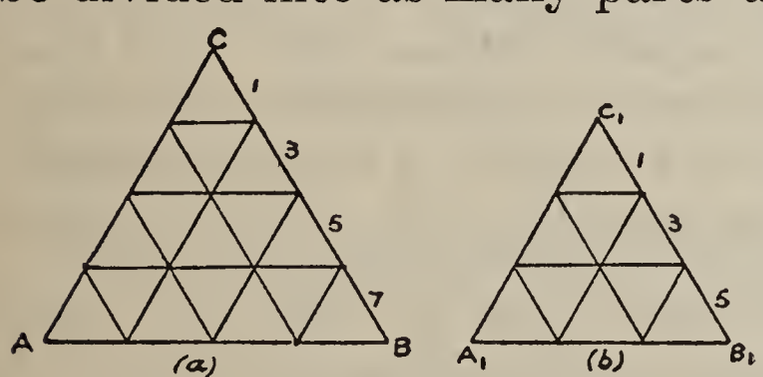

Fig. 40.

to each other as the squares of their corresponding sides. This is the most important theorem of plane geometry.

15. Ratio.—Two numbers, quantities or magnitudes are regarded as of the same kind when one of them can be said to be greater than, equal to, or less than the other.

The *ratio* of a magnitude to another of the same kind is the number expressing how much of the second is contained in the first. Thus, 3 apples are $\frac{3}{4}$ of 4 apples, so their ratio is as 3 to 4, written 3 : 4 or $\frac{3}{4}$. Ratio may, however, be also a new type of number. Thus, the Pythagorean Theorem says that the diagonal of a square is $\sqrt{2}$ when the side is 1.

If in Fig. 41 we attempt to divide $AB = 1$ into $AC = \sqrt{2}$, it will be contained once, as far as D, with remainder DC. Now draw DE at right angles to AC meeting BC at E, and connect AE. The two triangles ADE and ABE are right triangles with two sides AD and AE equal to AE and AB. They must therefore be equal triangles and, indeed, are two positions of the same triangle. This makes $BE = DE$, which also equals DC, for since D is a right angle and C is 45°, that is, half of a right angle being half of a "kite", then CED is also 45° and we know that equal sides are opposite equal angles. Notice also that the original square and the figure $ABED$ are kites.

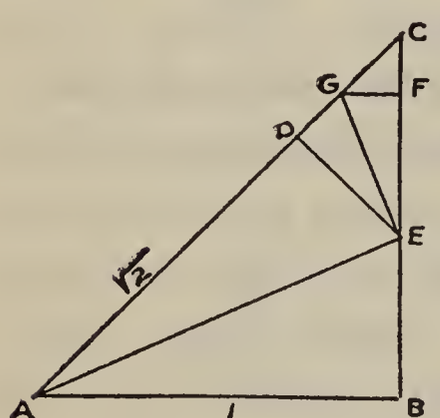

Fig. 41.

Arithmetic taught us that if AB and AC have a common factor, their difference DC must have this same factor. If, then, we treat the triangle EDC as we treated the same shaped or similar triangle ABC we shall find that FC must have this common factor and that we have a new triangle CFG similar to ABC. We might continue indefinitely obtaining such triangles without ever obtaining an expression for our ratio in the

numbers we are used to. Such ratios as $\sqrt{2}$ and 1 are called *incommensurable*, because they have no common measure or common factor. We learned in Arithmetic that most numbers obtained by measurement are of this class and we shall find that all our geometrical truths hold whether the ratios are expressed by commensurable or incommensurable numbers.

16. Proportion.—Proportion is a term applied to the equality of ratios: Thus $\frac{3}{4} = \frac{6}{8}$ is a proportion, sometimes written $3 : 4 = 6 : 8$ or $3 : 4 :: 6 : 8$. The numbers 3 and 6 are called *antecedents*, the numbers 4 and 8 are *consequents*. Also 3 and 8 are called *extremes*, while 4 and 6 are *means*.

A proportion can be generally expressed as $\frac{a}{b} = \frac{c}{d}$ where the letters represent four numbers, any three of which are given. In such a case the fourth number is readily determined, inasmuch as we know that the two ratios must be equal. If we multiply both sides of this proportion by bd we get $ad = bc$, for $\frac{a}{\cancel{b}} \times \cancel{bd} = \frac{c}{\cancel{d}} \times b\cancel{d}$. Hence, *the product of the means equals the product of the extremes.* Here a and d are evidently the extremes.

Numbers are sometimes repeated in a proportion as $\frac{a}{b} = \frac{b}{c}$, which gives $b^2 = ac$ and $b = \sqrt{ac}$. Here b is a *mean proportional* between a and c and equals the square root of their product. Also, c is called a *third proportional* to a and b, just as in $\frac{a}{b} = \frac{c}{d}$, d is called a *fourth proportional* to a, b and c. The proportion $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ is a *continued proportion.*

In any proportion the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent. This may be readily seen in the special case $\frac{3}{4} = \frac{6}{8} = \frac{9}{12}$, where

$\frac{3+6+9}{4+8+12} = \frac{18}{24} = \frac{3}{4} = \frac{6}{8} = \frac{9}{12}$. As proof of this, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each of these ratios equals some number $= r$.

If $\frac{a}{b} = r$, then $a = br$

If $\frac{c}{d} = r$, then $c = dr$

If $\frac{e}{f} = r$, then $e = fr$

Hence, $a + c + e = br + dr + fr = (b + d + f)r$,

therefore, $\frac{a+c+e}{b+d+f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

17. Various Forms of Proportions.—These proportions can be put into a variety of forms, the following of which are evident:

If $\frac{a}{b} = \frac{c}{d}$, then this is true upside down that is, $\frac{b}{a} = \frac{d}{c}$.

Also, if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$, for in both cases we have $ad = bc$ by taking the product of the means and the extremes.

The following theorem, or statement, is by the greatest of the ancient Greek geometricians, Archimedes:

Theorem: *If three or more parallel lines intercept equal segments on one transversal (a straight line intersecting other straight lines) they intercept equal segments on every transversal.*

Proof: If, as in Fig. 42, we draw the helping lines a'', b'', c'', parallel to the lines in which a, b, c, lie, we form parallelograms of which a and a'', etc., are opposite sides and are therefore equal; so a'', b'', c'' are all equal, being equal to equal things a, b, c. Hence the triangles 1, 2, 3, are equal, since in addition to the equal sides a'', b'', c'', they

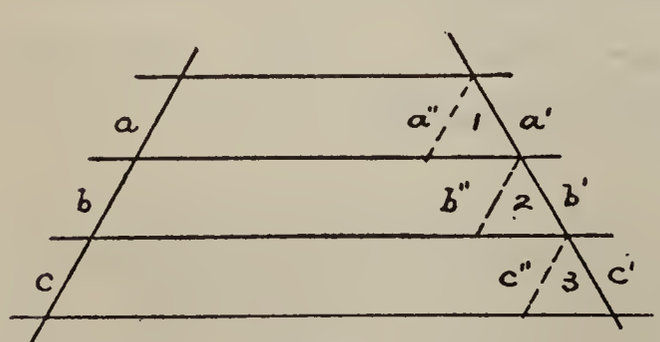

Fig. 42.

have two like placed angles made equal by the parallel lines.

Theorem: *If two parallels cut two intersecting transversals, the segments intercepted on one transversal are proportional to the corresponding segments on the other.*

In Fig. 43, the preceding theorem shows that if $\frac{CD}{DA} = \frac{M}{N}$, then $\frac{CE}{EB} = \frac{M}{N}$; therefore $\frac{CD}{DA} = \frac{CE}{EB}$. In this particular figure $m = 4$ and $n = 3$.

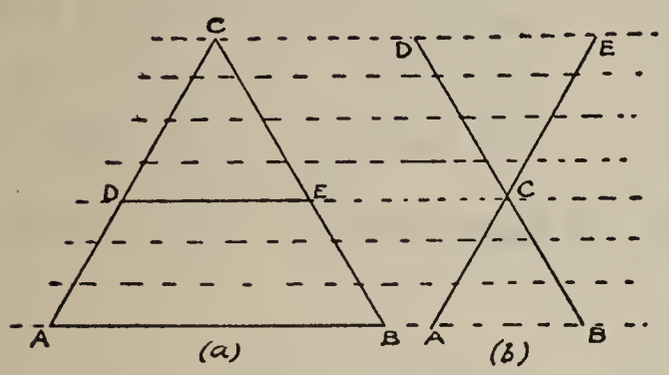

Fig. 43.

Here is another case of the same truth. Since this proportion can be written $\frac{CD}{CE} = \frac{DA}{EB}$ and since the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent, it follows that $\frac{CA}{CB} = \frac{CD}{CE} = \frac{DA}{EB}$.

It can be readily shown from Fig. 43 (*a*) and (*b*) that similar triangles, that is triangles having the same shape, can be distinguished by any of the following facts: (1) They have their corresponding angles equal, (2) they have their corresponding sides proportional, and (3) they have the sides about any equal angles proportional. The third fact was noted in Section 11 of this chapter.

18. Dividing Triangles.—*A right triangle is divided into similar triangles by a perpendicular from the right angle.*

We can easily prove that the smaller triangle *ADB*, Fig. 34, is similar to the whole triangle *ACB* because as each contains the angle *A* and a right angle, the angles *B* and *C* must be equal (the angles of any triangle being equal to two right angles) and therefore the triangles are similar, having the three angles equal. Likewise *CDB* can be proved similar to *ACB*.

Theorem: *The bisector of an angle of a triangle divides the opposite side in the ratio of the other two sides.*

Draw the helping line CE Fig. 44, parallel to the bisector DA, which meets BA (extended) at E. That the numbered angles are equal is evident. Now since $C_2 = E$, the triangle ACE is isosceles and $AC = AE$. From Fig. 36, we know $\frac{BD}{DC} = \frac{BA}{AE}$, and if we replace AE by its equal AC, we learn that $\frac{BD}{DC} = \frac{BA}{AC}$, which is what we wished to prove.

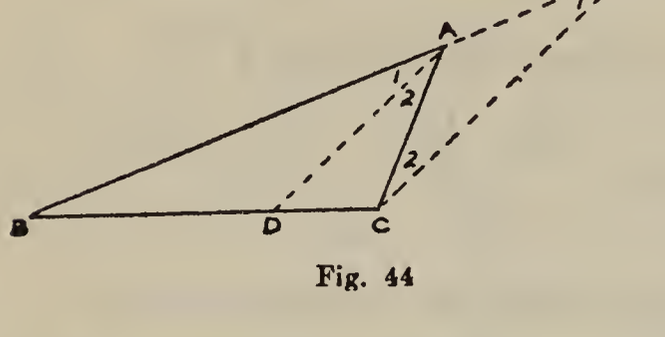

Fig. 44

19. Exterior Division of a Line.—The foregoing demonstration is applicable to Fig. 45, where the angle bisected is the exterior angle. Here we come upon a new conception of division of a line called *external division*, where BC is thought of as divided at D into the two parts BD and DC, just as a journey from Philadelphia to Pittsburgh might be thought of as two journeys—from Philadelphia to Chicago and from Chicago to Pittsburgh. When a line is divided internally and externally in the same ratio it is said to be divided *harmonically*. Evidently the bisection of the interior and exterior angle at the same vertex divides the opposite side harmonically. The reader should draw this to scale and verify the accuracy of the principle by measuring the lines carefully with his dividers. The *converse* of this theorem, that a line from a vertex of a triangle which divides the opposite side and cuts parts proportional to these other sides is a bisector, is also

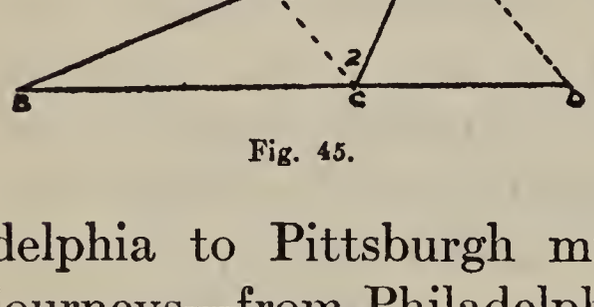

Fig. 45.

true and can be easily proved by reversing all the steps.

20. Constructions.—It is now evident that we can add and subtract lines by placing them end to end. It is also possible to multiply and divide lines geometrically. This is especially desirable when the lines are incommensurable. To multiply the lines AC and AD, Fig. 46, we lay off AB, equal to one unit, and produce it till it equals one of our lines, say AC. Now at any convenient angle lay off the other line AD, draw BD, and thru C draw CE parallel to BD meeting AD (produced) at E. Then AE is the product of AC and AD. From similar triangles $\frac{1}{AD} = \frac{AC}{AE}$, then since the product of the means equals the product of the extremes, $AE = AD \times AC$. To divide AE by AC, we construct $AB = 1$, and draw BD parallel to CE. The quotient is AD.

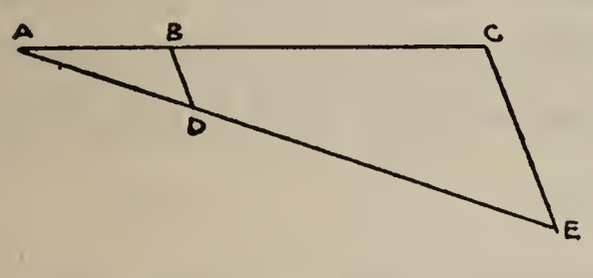

Fig. 46.

The first part of the construction shows us how to find a fourth proportional to any three lines AB, BC and AD. If $AC = AD$, we would obtain a third proportional to two lines.

To construct a mean proportional to the lines AB and BC, we place them end to end, as in Fig. 47, and regarding them as a diameter, we construct a circle. The length of the perpendicular from the circle to B is a *mean proportional* between AB and BC. This becomes clearer when we draw the helping (dotted) lines AD and DC. Then as all the right triangles are similar,

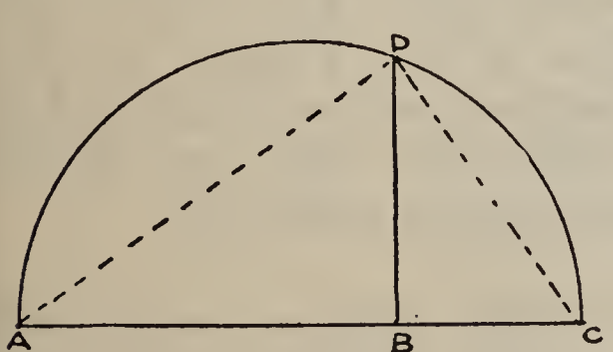

Fig. 47.

it follows that the two smaller right triangles have their sides proportional, that is, $\frac{AB}{BD} = \frac{BD}{BC}$, which shows that BD is a mean proportional to AB and BC.

21. Dividing a Line Into Mean Sections.—To divide a line into mean sections is to find the part which shall be a mean proportional to the whole line and the remaining part. This was called by the Greeks the Golden Section. In all properly made books, pictures, sections of house walls, etc., the short side is the mean section of the long side. The page of this book illustrates the principle.

As proof of construction, make BC of Fig. 48 $= \frac{1}{2}AB$ and draw a circle. Then draw the line $ADCE$ and draw DF parallel to EB and EG parallel to DB. We can show that BF is the *internal mean section* of AB, while BG is the *external mean section*.

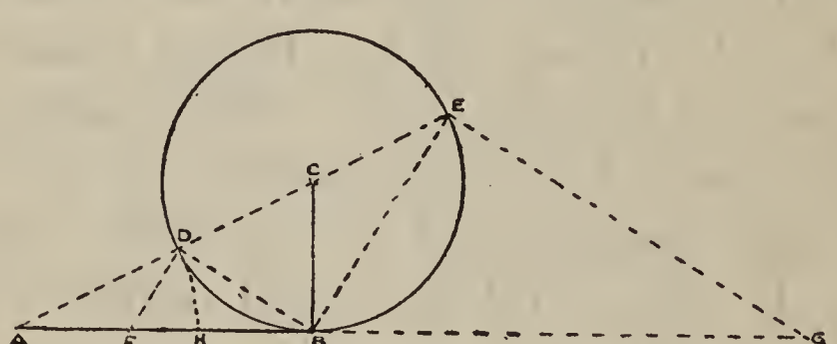

Fig. 48.

Remembering that $DE = 2\ (BC) = AB$, and that AB is a tangent to the circle, we see from the figure that $\frac{AD}{DE} = \frac{AD}{AB} = \frac{AB}{AE} = \frac{DE}{AE}$; hence DE or AB is a mean section of AE. Further, as AB is similarly divided, FB is the internal mean section of AB, and BG is the external. It can also be shown that AH (equal to AD) is also the mean section of AB and therefore equals FB.

Theorem:—*An isosceles triangle whose unequal side is the mean part of the equal sides, has its vertex angle one-half each equal angle and therefore one-tenth of four right angles.*

Proof: Make $AD = BC$, Fig. 49, and join DC. Since $AD = BC$ it is the mean part of AB, then $\frac{BD}{DA} = \frac{DA}{BA} = \frac{BC}{CA}$. Hence DC is a bisector of the angle C, (see section 18 of this chapter) and since the angle B is common to the triangles DBC and ABC and the sides $\frac{DB}{BC} = \frac{BC}{BA}$ (as BC is the mean part) the triangles are similar and have their angles equal. Therefore, Angle $A = \frac{1}{2}C = \frac{1}{2}B$; the sum of all the angles of the triangle equals $\frac{5}{2}B = 180°$, and A equals $\frac{1}{2}B = \frac{1}{5}$ of $180°$ or $\frac{1}{10}$ of $360°$.

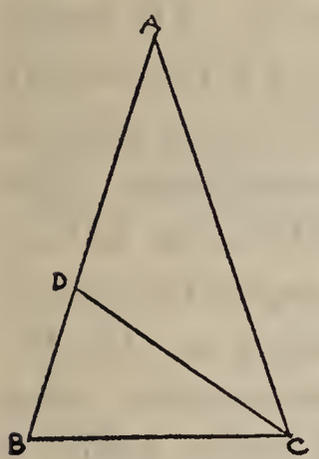

Fig. 49.

22. Similar Polygons.—Similar polygons are polygons whose corresponding sides are proportional and whose corresponding angles are equal.

Theorem: *Two similar polygons can be placed so as to be similarly situated about any chosen point.*

Make $OX = B_1A_1$ and parallel to BA in Fig. 50. Draw XM parallel to OB, MN parallel to AB, NP parallel to BC, etc., and the polygon $MNPQR$ is the polygon $A_1B_1C_1D_1E_1$, Fig. 50 placed to fit the conditions of the theorem.

In Fig. 51 is shown how to construct on a given side RS_1, a polygon similar to $RSTUV$, by drawing parallels. First draw diagonals from R to T and U respectively, then from S_1, draw a line S_1T_1 parallel to ST to the point at which it meets the diagonal RT. From that point T_1 draw a line T_1U_1 parallel to TU, and from U, a line U_1V_1 parallel to UV, and the construction is completed.

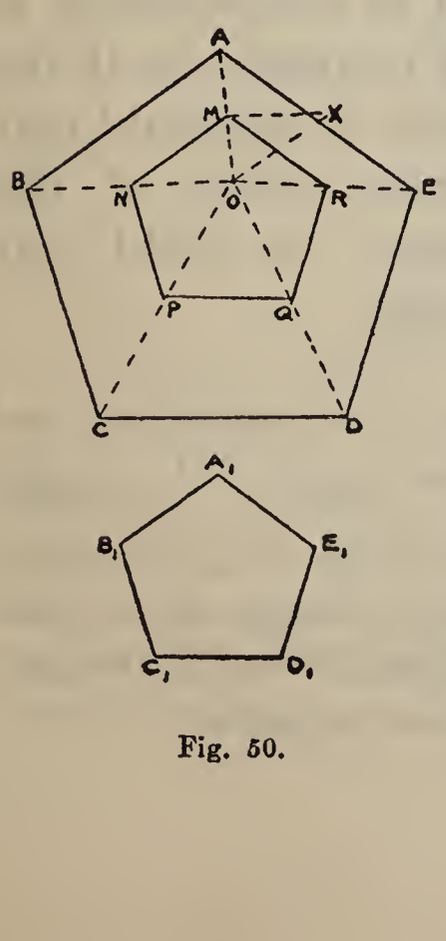

Fig. 50.

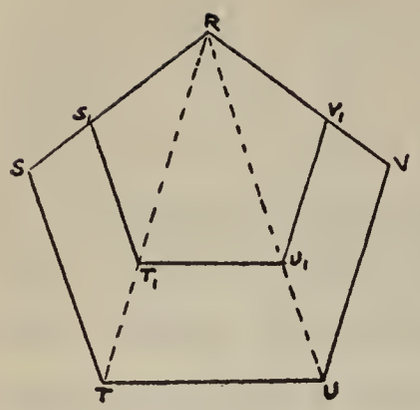

Fig. 51.

Since the square on AC, Fig. 52, is equal to the sum of the squares on AB and BC, and since the square on AD equals the sum of the squares on AC and DC, evidently the square on AD equals the sum of the squares on AB, BC and CD, and as similar figures are to each other as the squares of their homologous (like placed) sides, we can construct on AD as a side a polygon which will equal the sum of similar polygons constructed on AB, BC, and CD as corresponding sides. Likewise a circle whose radius is AD equals the three circles whose radii are AB, BC, and CD.

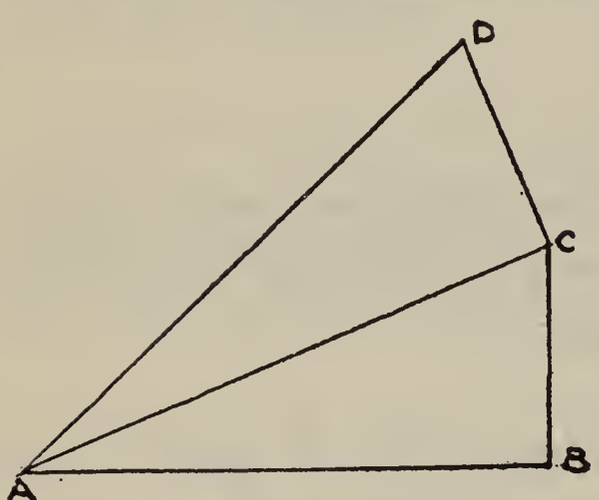

Fig. 52.

23. Regular Polygons.— Regular polygons have their sides equal and their angles equal. It can easily be proved that regular polygons of the same number of sides are similar. It is evident on drawing one, as Fig. 53, that it may be inscribed in a circle and is composed of as many isosceles triangles as it has sides. Evidently if we could construct the vertex angle of this isosceles triangle, we could construct the polygon.

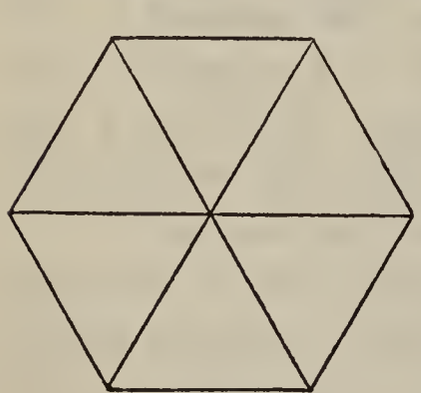

Fig. 53.

As each angle of an equiangular and therefore equilateral triangle is $\frac{1}{3}$ of $180° = 60° = \frac{1}{6}$ of $360°$, we can construct a regular hexagon in a circle by using the radius as a chord. We could construct the regular triangle in the same circle by taking every other vertex. Polygons of twelve, twenty-four, etc., sides could be constructed by bisecting arcs.

The square can be constructed by drawing two perpendicular diagonals and connecting their ends. Polygons of eight, sixteen, etc., sides are easily constructed by bisecting arcs.

By constructing a mean section of the radius and using it as a chord we get an isosceles triangle whose vertex angle is $\frac{1}{10}$ of 360°. Hence the mean section of a radius is the side of a decagon (ten-sided polygon) in the same circle. Polygons of five, twenty, forty, etc., sides can then be constructed.

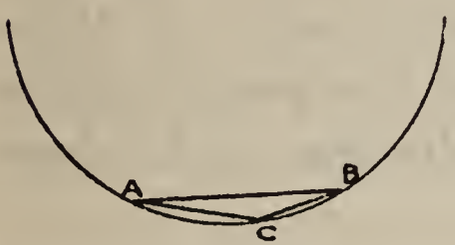

Fig. 54.

Since $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$ we can construct a fifteen-sided polygon by using chord *BC*, Fig. 54, as a side, where *AB* is the side of a hexagon and *AC* a side of a decagon.

The eminent mathematician Carl Frederick Gauss when 19 years of age constructed with ruler and compass a regular polygon of 17 sides and proved its correctness, but the proof is extremely difficult and need not be given here.

These are all the types of regular polygons constructible by ruler and compasses.

The altitude of each equal isosceles triangle in a regular polygon is called the *apothem*. As the area of a triangle equals one-half the product of its base by its altitude, if we call the sum of the sides of the polygon, that is, the sum of the bases of these triangles, the *perimeter*, we will have the area of a regular polygon equal to $\frac{1}{2}$ the perimeter times its apothem. In Fig. 55 *AB* is the apothem.

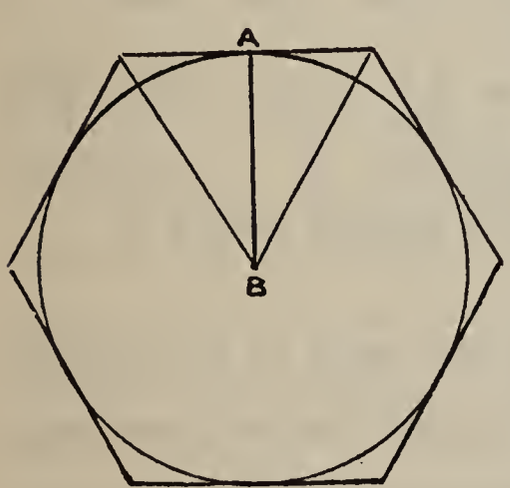

Fig. 55.

24. Area of a Circle.—As the number of sides of a regular polygon is increased and it assumes nearer and

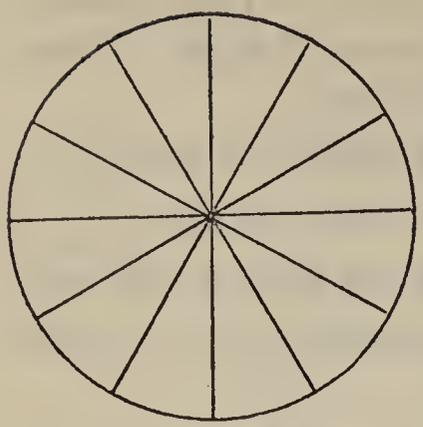

Fig. 56.

nearer the form of a circle, the perimeter of the polygon approaches the circumference, and as the apothem is the radius we say *the area of a circle is one-half the circumference multiplied by the radius.* If the sectors, Fig. 56, were taken from a circle and fitted as in Fig. 57, the more there were of these sectors the more they would resemble a parallelogram whose base was half the circumference and whose altitude was the radius.

Fig. 57.

The perimeters of any two regular polygons of the same number of sides must be to each other as their apothems, since homologous parts of similar figures are in proportion. This is also true of circles. If C_1 and C_2 represent circumferences of two circles, and D_1 and D_2 their diameters, we have $\frac{C_1}{C_2} = \frac{D_1}{D_2}$, which from the theory of proportion may be written $\frac{C_1}{D_1} = \frac{C_2}{D_2}$, hence the ratio of the circumference of the diameter is a constant, called in modern terms *pi* from the Greek letter π.

Archimedes calculated the value of the ratio to be between $3\frac{1}{7}$ and $3\frac{10}{71}$. It has been worked out to 707 decimals and proof has been furnished that its value cannot be found exactly as a decimal. By taking sides of inscribed and circumscribed polygons of 4, 8, 16, etc., we can find by long and careful tho not difficult computations that with a diameter equal to 1, the length of the perimeter of the inscribed polygon of 512 sides is 3.14137, while that of the circumscribed polygon is 3.14163, showing that

3.1416, which is generally used, is a close approximation. A closer value, and one easy to remember, is the fraction $\frac{355}{113}$.

25. Loci of Points and Sets of Lines.—We have learned that in a plane the number of straight lines and points is uncountable. It is, however, a more restricted statement to say that a point is on a line, than to say that a point is in a plane. It reduces the statement from two dimensions, length and breadth, to one—length. Now if we say that the point also lies in another given straight line, its position is completely determined, for two straight lines have but one point in common.

Reciprocally, a line in a plane gives a choice in two dimensions, but if the line passes thru a given point in the plane, we are restricted to the *pencil* of lines thru that point, as in Fig. 58. Our choice is now in one dimension—length. If we further know that the line passes thru a second point, it must be on a second pencil thru that second point. There is only one line common to these two pencils, for two points have only one straight line in common.

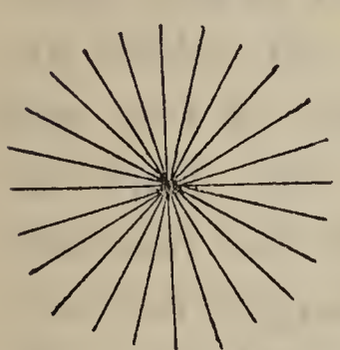

Fig. 58.

Of course, two straight lines may coincide, when they would have all their points in common, or two points may coincide, when they would have each line in the pencil in common. In general, however, two conditions leave us only a finite or countable choice of points or lines. For example, if it is required that a point shall lie in a given line and at a given distance from a given point, it is evident there are only two points, one on either side of the given point, which satisfy the conditions.

If instead of the condition we think of the figure

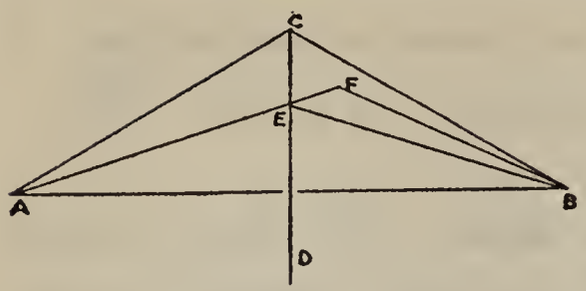

Fig. 59.

which replaces it, we call this figure in one case a *locus of points*, and in another a *set of lines*, satisfying the given condition. Evidently the perpendicular bisector *CD* of the line *AB*, Fig. 59, is the locus of points equidistant from *A* and *B*, for we know that every point in this perpendicular bisector is equidistant from *A* and *B*, that every point equidistant (as *E*) is on the perpendicular bisector, *or* that every point not on the perpendicular bisector (as *F*) is not equally distant.

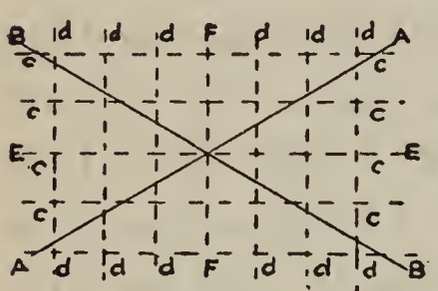

Fig. 60.

It is equally true that the set of lines equally inclined to two given lines, *AA*, *BB*, Fig. 60, which are not parallel, consists of two sets of parallels, *cc* and *dd*. Each set is parallel to one of the bisectors *EE* and *FF* of the angles between the two lines, hence the sets are perpendicular to each other. It also can be shown easily that the bisectors of the angles between two lines form the locus of points equally distant from the two lines. If the two lines were parallel, the locus would be a single line parallel and mid-way between them, as shown in Fig. 61.

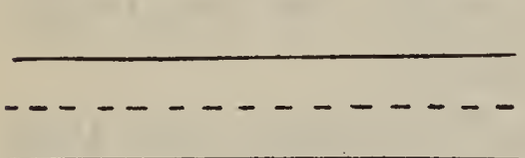

Fig. 61.

Let us now find the set of lines which have equal distances from two given points. If the two points are to lie on the same side of the lines, the sets are parallel to the line *AB*, Fig. 62. If *A* and *B* are to lie on opposite sides of the lines, the lines are a pencil of

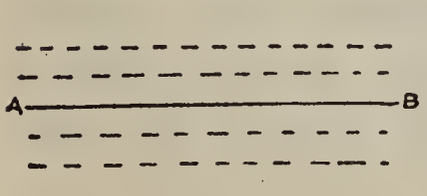

Fig. 62.

lines thru the mid-point of the line AB, as shown in Fig. 63. There are two sets of lines (dotted in Figs. 61 and 62) to be considered, just as there are two distinct lines which constitute the locus of points in Fig. 60.

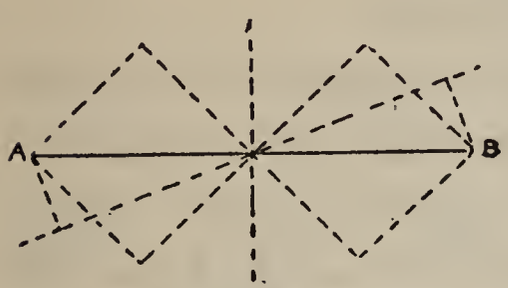

Fig. 63.

26. Intersection of Loci.—We noticed that two conditions were necessary to locate or determine, finitely, a point or a line. The points which shall be at equal distances from two given points A and B, in Fig. 64, and also at equal distances from two given lines aa and bb, will evidently be where the perpendicular bisector of AB meets the bisectors of the angles between the lines aa and bb. There are in general two points x, x, where the two loci cross each other. But the perpendicular bisector of AB might be parallel to one of the bisectors of the angles between the lines aa, bb, hence could cut only one of them, and there would be one point x. If, however, the perpendicular bisector of AB coincided with one of the bisectors of the angles between the lines aa, bb, there would be an uncountable number of points x.

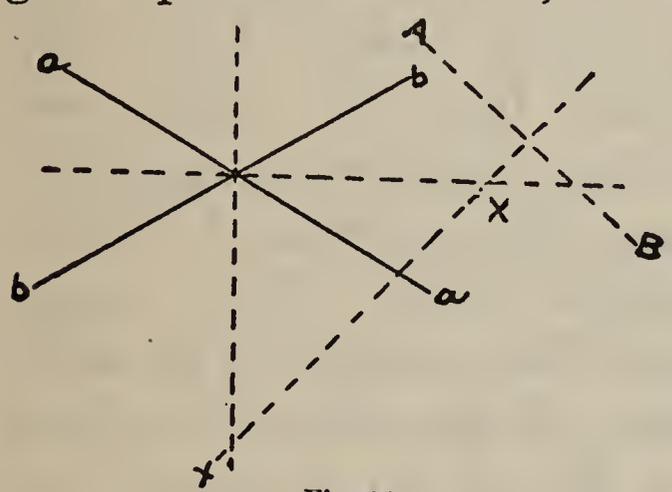

Fig. 64

Lines which are equally inclined to two given lines aa and bb, Fig. 65, and are at equal distances from two given points A and B, are evidently parallel to the bisectors of the angles between aa and bb and coincide with

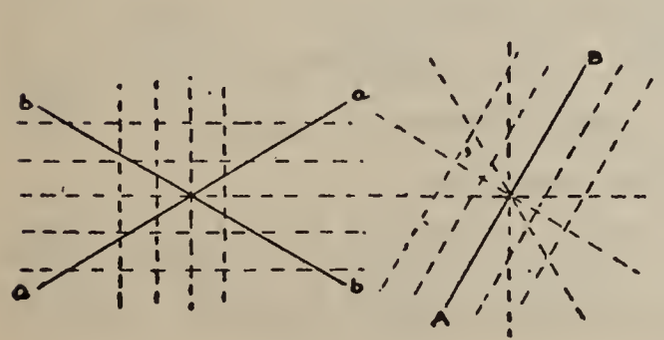

Fig. 65.

either the lines parallel to AB or the pencil thru the mid-point of the line AB. There are in general two lines, but only one is shown in the figure, the other being left to the ingenuity of the reader.

27. The Circle as Locus.—A circle constructed by compasses can evidently be considered as the locus of a point at a given distance from a given point. From Fig. 66, the locus of the mid-points of a set of parallel chords is a diameter AB bisecting the arcs and the angles at the center subtended by the chords. It is also perpendicular to the chords. Evidently equal chords are equally distant from the center and are tangent to a circle whose radius is their common distance from the center of the original circle. This second circle is called *the envelope of the set of chords*. If the ruler AB be fixed as in Fig. 67 to a center C, it will trace out a circle as illustrated on a plane strewn with sand and illustrate an "envelope".

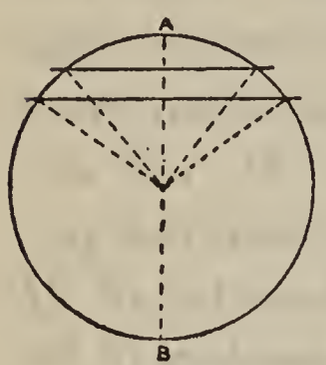

Fig. 66.

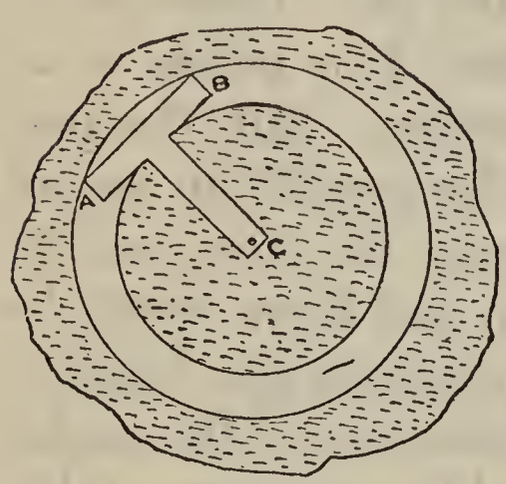

Fig. 67.

If a real-estate dealer is asked to purchase a lot one mile from a straight railroad and four miles from a town which is two miles distant from this railroad, he has four choices of location as shown at x, x, x, x in Fig. 68. An endless variety of similar problems is open to the ingenious reader.

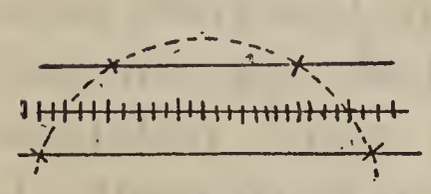
Fig. 68.

In Fig. 69 we add to the "kite" $OBPA$ the links AQ, QB, RQ, all of which are made equal to OR, which

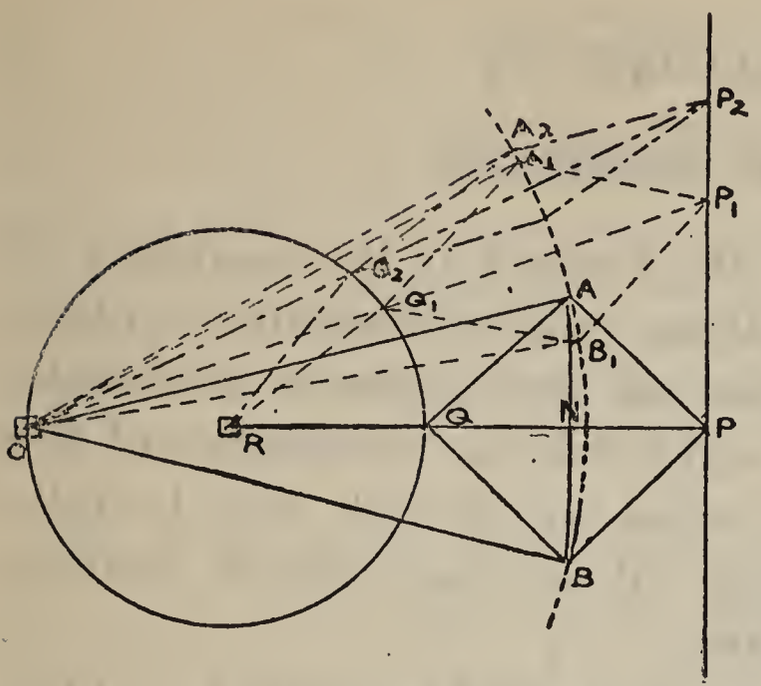

Fig. 69.

itself is equal to AP or PB. When O and Q are fixed, Q will move on a circle and the angle OQ_1Q will be a right triangle and similar to OPP_1 if PP_1 is perpendicular to OP. This makes $OQ_1 : OQ = OP : OP_1$ whence $OQ_1 \times OP_1 = OQ \times OP$.

Now $(OA)^2 = (ON)^2 + AN^2$

$(AP)^2 = (PN)^2 + AN^2$

Therefore $(OA)^2 - (AP)^2 = (ON)^2 - (PN)^2 =$

$(ON - PN)(ON + PN) = OQ \times OP.$

And since OA and AP are given lengths, $OQ \times OP$ is always the same and P must move so that PP_1 is perpendicular to OP and hence the moving point P has a straight line for its locus. All the parts of the figure can be measured by compasses and dividers, and so for the first time we are actually able to draw a straight line without copying.

REVIEW

1. What is a trapezoid? A trapezium?
2. Give a brief description of the Great Theorem of Pythagoras.
3. What does "similar" mean? Explain your answer.
4. State and explain the relationship between ratio and proportion.
5. What is a "pencil of lines"? A locus of points?
6. How do you obtain the "envelope" of a set of equal chords?

CHAPTER VI

SOLID GEOMETRY

1. Lines and Planes.—We learned at the beginning of Plane Geometry that three points determine a plane and that the intersection of two planes is a straight line. We spoke of straight lines as boundaries of the parts of a plane, but when we do not wish to take note of the planes being cut we can refer to straight lines as lying in the plane.

There is evidently an uncountable number of lines thru each point of a plane. A line is said to be perpendicular to a plane if it is perpendicular to every one of the lines in the pencil thru the point where it meets the plane. This definition can be much simplified, as follows:

Theorem: *If a straight line is perpendicular to each of two other straight lines at their point of intersection it is perpendicular to the plane of those lines.*

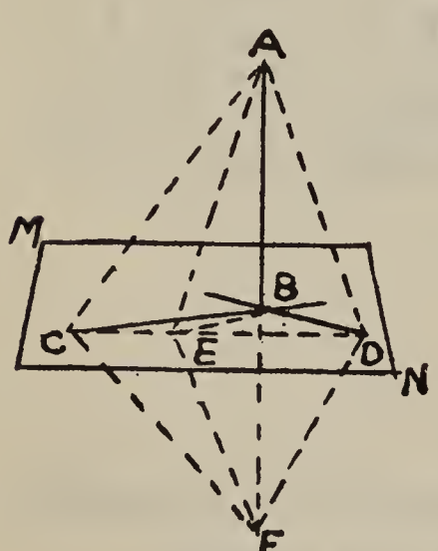

Fig. 70.

Let AB in Fig. 70 be perpendicular to CB and DB at the point B Now prove that AB is also perpendicular to EB or any other line thru B in the plane MN of these two lines CB and DB.

Proof: Draw any lines in the plane MN cutting these three lines CB, EB, and BD in the points C, E, and D. Now produce AB its own length to F, and join A and F to the three points C, E, D. Notice that tho the whole figure is a solid, it is bounded by planes. Now $AC = FC$ since the right triangles ABC and FBC are equal; similarly $AD = FD$. This makes the triangles

ACD and *FCD* equal, because their three sides are equal; hence the two *angles ACE* and *FCE* are equal. As the two *triangles ACE* and *FCE* are equal, having two sides and the included angle equal, it follows that $AE = FE$. We now have the two *triangles ABE* and *FBE* equal because their three sides are equal; therefore, the *angle ABE* = the angle *FBE*. This by plane geometry makes *EB* a perpendicular since the two triangles are in the same plane. Now as *AB* is perpendicular to any line *EB*, it is perpendicular to every line thru *B*, as we had to prove.

We could prove that all perpendiculars to a given line at a given point lie in a plane perpendicular to the line at that point and that there can be but one plane thru a given point perpendicular to a given line. It is also true, as in plane geometry, that the perpendicular from a point to a plane is the shortest line and that equal oblique lines meet the plane at equal distances from the foot of the perpendicular.

2. Dihedral Angles.—The angle between two intersecting planes is a *dihedral.* The intersection of the two planes is called the *edge of the dihedral angle.* This is measured by the plane angle formed by perpendiculars in each plane to the edge at the same point. Dihedral angles are classified as acute, right, adjacent, supplementary, etc., in the same manner as plane angles. Evidently a plane passed thru a line perpendicular to another plane will form a right dihedral angle and the planes will be perpendicular to each other. This plane might be the particular one determined by the perpendicular and any other line either parallel or oblique to the plane. It is evident that there is but one plane determined by this perpendicular and the other line, hence only one plane perpendicular to another plane can be passed thru any line not perpendicular to the plane. The intersection of these two planes is called

the *projection of the oblique line on the plane.* The angle a line makes with its projection is the least angle it makes with the plane and is defined as the *angle with the plane.*

Theorem: *If a line in a plane is perpendicular to the projection of a line where the line and its projection meet, it is also perpendicular to the line.*

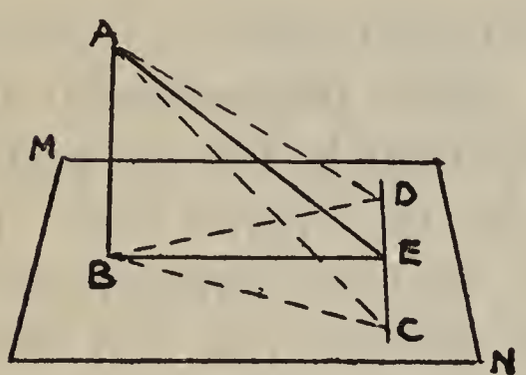

Fig. 71.

We want to prove that if *CD* in Fig. 71 is perpendicular to *BE*, the projection of *AE*, it is also perpendicular to *AE*. If we make *CE* = *DE* and draw the other lines of the figure, *BC* will equal *BD*, and because of this *AC* will equal *AD*. This makes the *angles DEA* and *CEA* equal and therefore right angles, because of the equality of the *triangles DEA* and *CEA* which have three sides equal.

Theorem: *Two straight lines perpendicular to the same plane are parallel.*

If we draw *FE* thru *D* perpendicular to *BD* in Fig. 72, then *AD* will be perpendicular to *FE* by the preceding theorem. And as

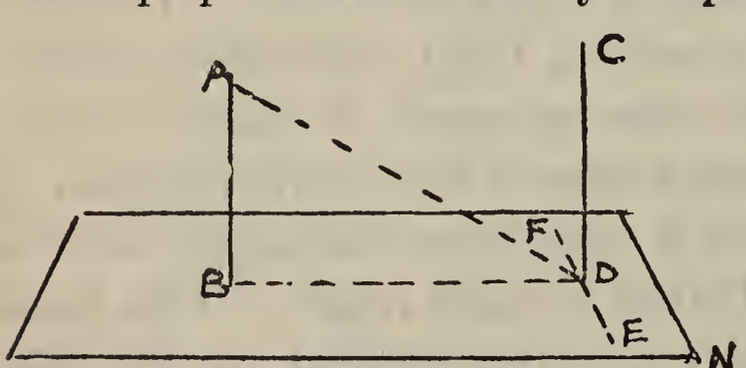

Fig. 72.

CD was perpendicular to the plane it is by definition perpendicular to *FE*. As all perpendiculars to the same line at the same point are in a plane, the three lines *BD*, *AD*, and *CD* are in a plane. Also, as *AB* joins two points of this plane, it is in this same plane, and *AB* and *CD* are two perpendiculars to the same line and by plane geometry are parallel.

3. Parallel Planes.—A line and a plane or two planes are said to be parallel if they will not meet when ex-

tended. Evidently a plane containing only one of two parallel lines is parallel to the other. There are theorems about parallel planes corresponding to these about parallel lines.

Theorem: *The intersections of two parallel planes by a third plane are parallel lines.*

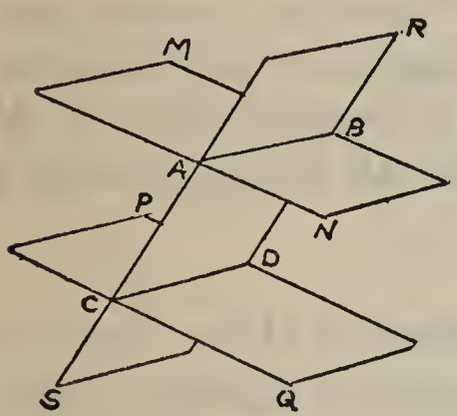

Fig. 73.

Evidently AB and CD, Fig. 73, are in the same plane SR and cannot meet, since they are in the parallel planes MN and PQ. Hence they are parallel.

Theorem: *If two straight lines are intersected by three parallel planes, their corresponding segments are proportional.*

Let AB and CD, Fig. 74, be intersected by the parallel planes MN, PQ, RS, in the points A, E, B and C, F, D.

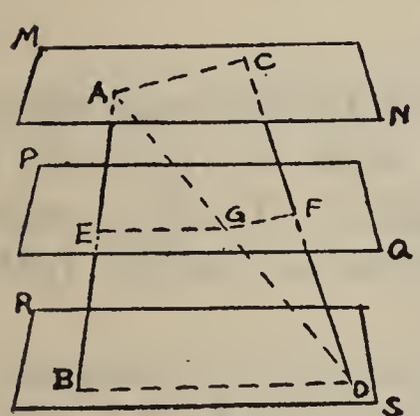

Fig. 74.

To prove that $\frac{AE}{EB} = \frac{CF}{FD}$ draw AD cutting the plane PQ in G and draw AC, BD, EG and FG. Then EG is parallel to BD, and GF is parallel to AC. Therefore $\frac{AE}{EB} = \frac{AG}{GD}$ and $\frac{CF}{FD} = \frac{AG}{GD}$. Consequently, $\frac{AE}{EB} = \frac{CF}{FB}$.

Theorem: *Between two straight lines not in the same plane, there can be one common perpendicular and only one.*

Let DC and AB, Fig. 75, be the two lines If AG is drawn parallel to DC it will, with AB, determine a plane MN, and if the

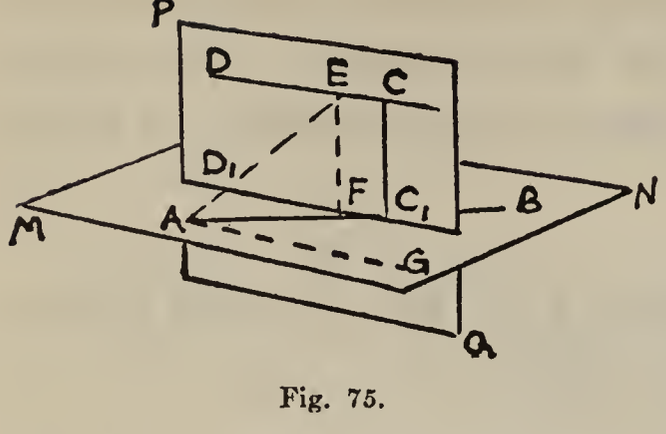

Fig. 75.

plane PQ is passed perpendicular to MN it will intersect MN in D_1C_1, C_1 being on AB. Now draw C_1C perpendicular to MN and it will also be perpendicular to AB and D_1C_1. As CC_1 is evidently in PQ, it must be perpendicular to DC, which is parallel to D_1C_1, and therefore is the perpendicular required. No other common perpendicular such as EA can be drawn, for any such perpendicular would have to coincide with EF, which is perpendicular to M and does not pass thru C_1, the only point of AB in the intersection of the two planes.

4. Polyhedral Angles.— The opening of three lines or planes which meet at a common point is called a *polyhedral angle.* The corner of a room is generally a *trirectangular trihedral angle*—trihedral because three planes meet and trirectangular because the plane angles are right angles.

If the faces of a polyhedral angle S–$ABCD$, Fig. 76, are produced thru the vertex S, another polyhedral angle S–$A_1B_1C_1D_1$ is formed, symmetrical with respect to S–$ABCD$. The face angles ASB, BSC, etc., are equal, respectively, to the face angles A_1SB_1 B_1SC_1, etc.

Also the dihedral angles SA, SB, etc., are equal, respectively, to the dihedral angles SA_1, SB_1, etc. (The second figure shows a pair of vertical dihedral angles.)

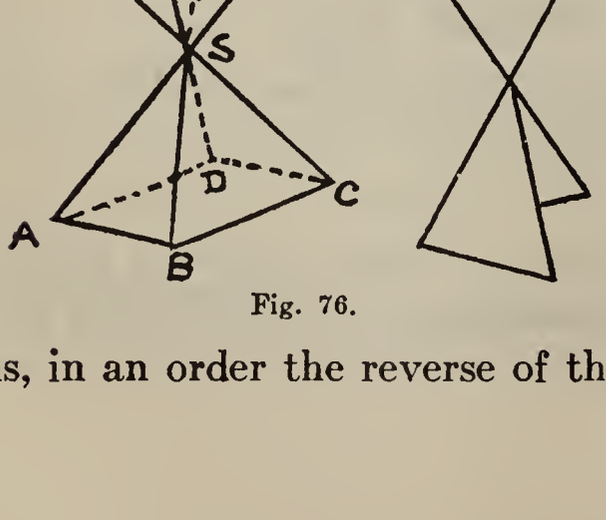

Fig. 76.

The edges of S–$ABCD$ are arranged from *left to right* (counter clockwise) in the order SA, SB, SC, SD, but the edges of S–$A_1B_1C_1D_1$ are arranged from *right to left* (clockwise) in the order SA_1, SB_1, SC_1, SD_1—that is, in an order the reverse of the order of the edges in S–$ABCD$.

Two symmetrical polyhedral angles, therefore, have all their parts equal, each to each, but arranged in reverse order.

Theorem: *The sum of any two face angles of a trihedral angle is greater than the third face angle.*

If the face angle ASC, Fig. 77, is greater than ASB or BSC it is sufficient to prove their sum greater than ASC. We will construct the angle ASD equal to ASB and take $SD = SB$ and pass the plane ABC thru B and D; then the *triangle* ASD equals ASB and AD equals AB, and as AC is less than $AB + BC$, so, taking away equals, DC is less than BC. As the triangles DSC and BSC have two sides equal, but DC less than BC, the angle DSC is less than BSC. Adding the equal angles ASD and ASB we have ASC less than $ASB + BSC$.

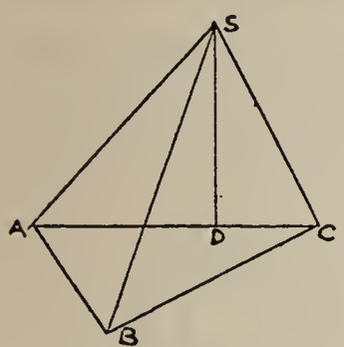

Fig. 77.

Theorem: *The sum of the face angles of any convex polyhedral angle is less than four right angles.*

Let S, in Fig. 78, be a convex polyhedral angle, and let all its edges be cut by a plane, making the section $ABCDE$.

To prove that the angle ASB plus the angle BSC, etc., is less than four right angles, first from any point O within the polygon draw OA, OB, OC, OD, OE. The number of the triangles having the common vertex O is the same as the number having the common vertex S. Therefore, the sum of the angles of all the triangles having the common vertex S is equal to the sum of the angles of all the triangles having the common vertex O.

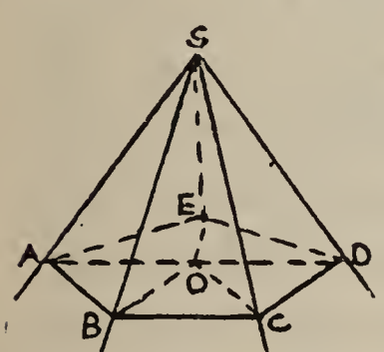

Fig. 78.

But in the trihedral angles formed at A, B, C, etc., the angle SAE plus the angle SAB is greater than the angle EAB, and the angle SBA plus the angle SBC is greater than the angle ABC, etc., by the previous theorem.

Hence, the sum of the angles at the bases of the triangles whose common vertex is S is greater than the sum of the angles at the

bases of the triangles whose common vertex is *O*. Therefore, the sum of the angles at the vertex *S* is less than the sum of the angles at the vertex *O*. But the sum of the angles at *O* is equal to four right angles. Therefore, the sum of the angles at *S* is less than four right angles.

5. Prisms, Parallelopipeds and Other Polyhedrons.—A *polyhedron* is a solid bounded by planes and is convex if no face produced will again intersect the polyhedron. When a polyhedron has four faces it is called a *tetrahedron;*

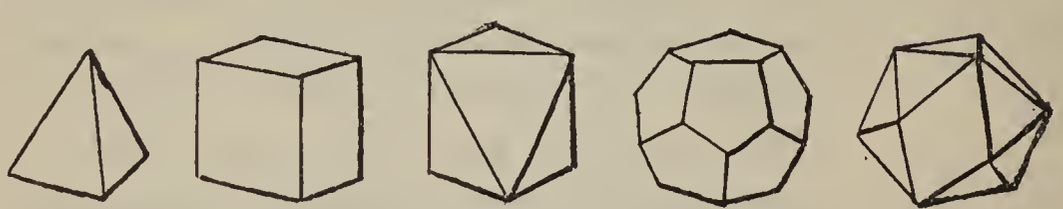

Fig. 79.

one of six faces, a *hexahedron*; one of eight faces, an *octahedron*; one of twelve faces, a *dodecahedron*; one of twenty faces, an *icosahedron*. The five solids in Fig. 79 are all the regular polyhedrons possible, since the sum of the faces at each vertex must be less than four right angles.

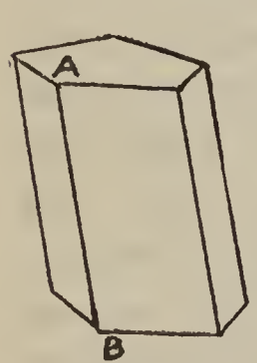

Fig. 80.

A *prism*, illustrated in Fig. 80, is a polyhedron two of whose faces are equal polygons in parallel planes and whose sides are opposite sides of the parallelograms which form its other faces. When the parallelograms are rectangles we have a right prism whose lateral edges *AB* are perpendicular to its bases. When all the faces are parallelograms we have a *parallelopiped*, as shown in Fig. 82, and when all the faces are rectangles we have a rectangular parallelopiped Fig. 83. If these rectangles are squares we have a *cube*,

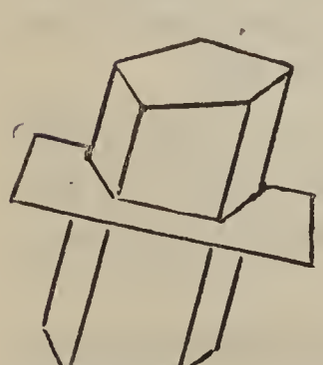

Fig. 81.

Fig. 84. A right section of a prism is made as shown in Fig. 81 by a plane perpendicular to its edges,

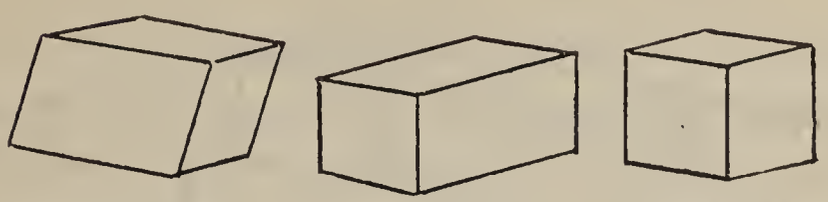

Fig. 82 Fig. 83 Fig. 84

As the faces of a prism are parallelograms, the perimeter of the right section is the sum of the altitudes of these parallelograms and the lateral area is equal to the product of this perimeter by a lateral edge.

Theorem: *An oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.*

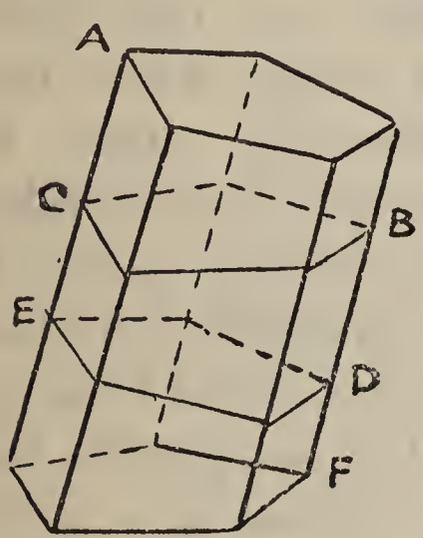

Fig. 85.

In Fig. 85 the top and bottom portions *AB* and *EF* are exactly equal, but the two upper parts *AB* and *CD* form a right prism and the two lower parts *CD* and *EF* are an oblique prism; hence the theorem.

By means of this theorem we can prove that a plane passed thru two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms as illustrated in Fig. 86.

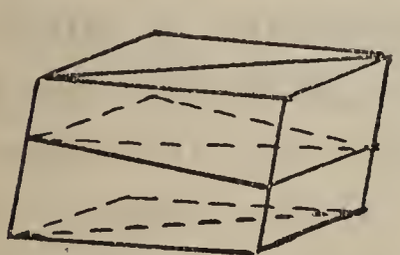

Fig. 86.

We assumed from cubic measure in arithmetic that the volume of a rectangular parallelopiped was equal to the product of its three dimensions, and it is shown in Fig. 87 that any parallelopiped is equivalent to a rectangular parallelopiped with the same base and altitude. In the figure

B, C and D are equal and their altitudes are the same. Therefore the volume of each equals the base by the altitude.

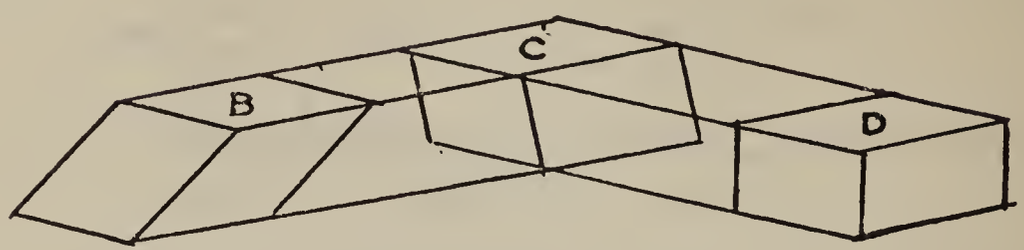

Fig. 87.

As the triangular prism in Fig. 86 has half the base and half the volume, its volume is also equal to the product of its base by its altitude and as all prisms can be divided into triangular prisms as shown in Figs. 88 and 89, the same is true of all prisms.

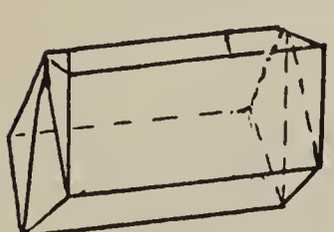

Fig. 88.

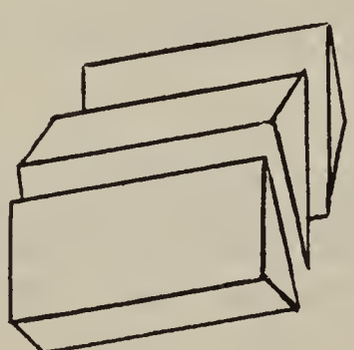

Fig. 89.

As a prism can have any number of sides, a cylinder can be regarded as the limit of a prism when the number of sides is increased indefinitely. The lateral area of a cylinder equals the circumference multiplied by the altitude, and the volume equals the area of the base by the altitude. As the base is a circle, the formula for lateral area is $S = 2\pi rh$, and for volume $V = \pi r^2 h$, where r = radius of base and h = altitude.

6. Pyramids.—A *pyramid* is a polyhedron of which one face called the base is a polygon, while the other faces are triangles meeting at a common vertex and whose bases are the sides of this polygon. If the polygon is regular and the perpendicular from the vertex meets its center, the pyramid is regular and the perpendicular is called the *axis*. The portion of a pyramid between

its base and a section cutting all the lateral edges is a *truncated pyramid* and when the section is parallel to the base this portion is called a *frustrum.* This frustrum is evidently the difference between two pyramids. As the faces of a regular pyramid are isosceles triangles and the faces of the frustrum of a regular pyramid are isosceles trapezoids, the lateral area is obtained by multiplying half the slant height by the sum of the perimeters of the bases.

Theorem: *If a pyramid is cut by a plane parallel to the base:*

(1) *The edges and altitude are divided proportionally.*

(2) *The section is a polygon similar to the base.*

(3) *The section is to the base as the square of the distance from the vertex is to the square of the altitude of the pyramid.*

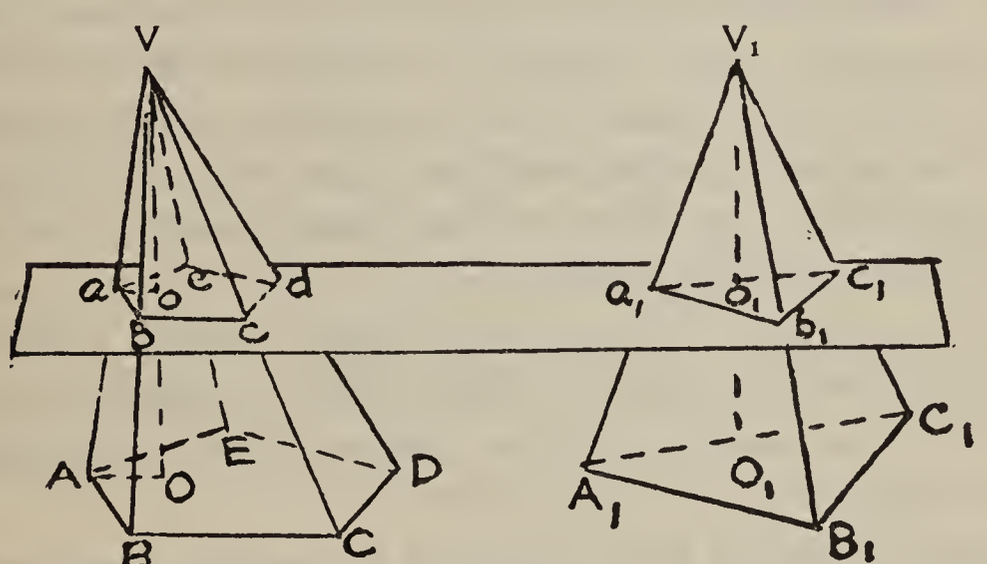

Fig. 90.

Referring to Fig. 90, (1) if straight lines are intersected by parallel planes their corresponding segments are proportional.

$$\frac{Va}{VA} = \frac{Vb}{VB} = \frac{Vc}{VC} = \text{etc.}$$

(2) By similar triangles

$$\frac{ab}{AB} = \left(\frac{Vb}{VB} = \frac{bc}{BC}\right) = \left(\frac{Vc}{VC}\right) = \frac{Cd}{CD} = \text{etc.}$$

Dropping parentheses, homologous sides are proportional, also homologous angles are equal as sides are parallel. Therefore polygons are similar.

$$(3) \qquad \frac{abc}{ABC} = \frac{ab^2}{AB^2} = \frac{Va^2}{VA^2} = \frac{Vo^2}{VO^2}$$

Theorem: *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*

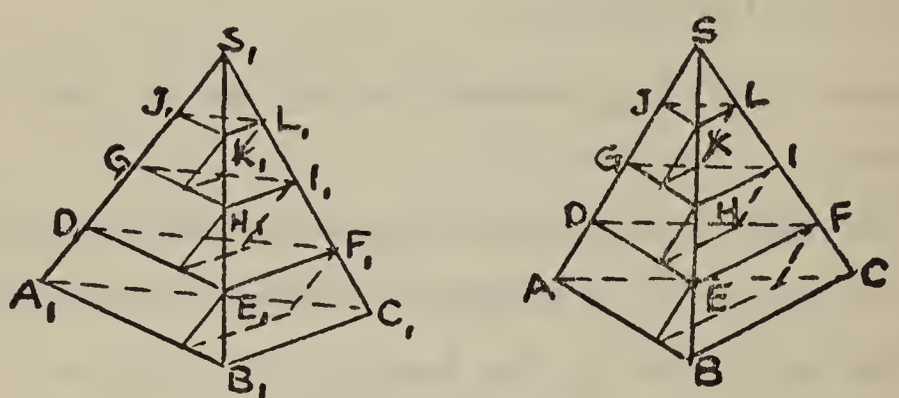

Fig. 91.

Let S–ABC and S_1–$A_1B_1C_1$, Fig. 91, be two triangular pyramids having equivalent bases situated in the same plane, and equal altitudes. To prove that S–ABC equal S_1–$A_1B_1C_1$, divide the altitude into n equal parts, and thru the points of division pass planes parallel to the plane of the bases, forming the sections DEF, GHI, etc., $D_1E_1F_1$, $G_1H_1I_1$, etc.

In the pyramids S–ABC and S_1–$A_1B_1C_1$ inscribe prisms whose upper bases are the sections DEF, GHI, etc., $D_1E_1F_1$, $G_1H_1I_1$, etc. The corresponding sections are equivalent, therefore, the corresponding prisms are equivalent.

Denote the sum of the volumes of the prisms inscribed in the pyramids S–ABC by V, and the sum of the volumes of the corresponding prisms inscribed in the pyramid S_1–$A_1B_1C_1$ by V_1. Then V is equal to V_1.

Now let the number of equal parts into which the altitude is divided be indefinitely increased. The volumes V and V_1 are always equal, and approach as limits the pyramids S–ABC and S_1–$A_1B_1C_1$, respectively.

Hence S–ABC is equal to S_1–$A_1B_1C_1$.

Theorem: *The volume of a triangular pyramid is equal to one third of the product of its base by its altitude.*

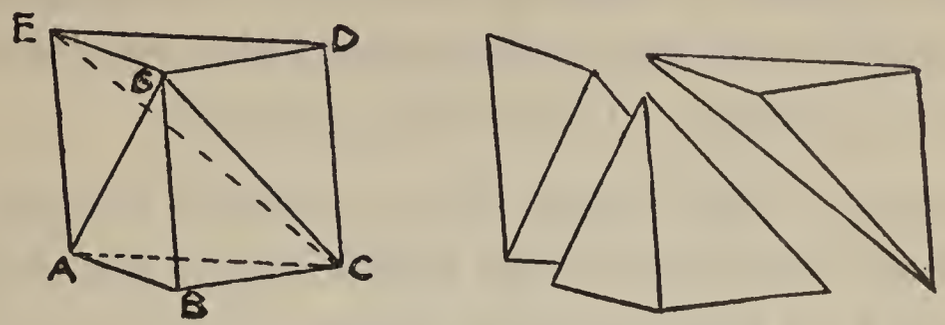

Fig. 92.

Let V denote the volume, B the base, and H the altitude, of the triangular pyramid S–ABC, Fig. 92. To prove that $V = \frac{1}{3}B \times H$, on the base ABC construct a prism ABC–ESD, having its lateral edges equal and parallel to SB. The prism is composed of the triangular pyramid S–ABC and the quadrangular pyramid S–$ACDE$. Thru SE and SC pass a plane SEC. This plane divides the quadrangular pyramid into the two triangular pyramids S–ACE and S–DEC, which have the same altitude and equal bases, as EC is a diagonal of a parallelogram. Therefore S–ACE is equal to S–DEC.

The pyramid S–DEC may be regarded as having ESD for its base and C for its vertex, and is, therefore, equivalent to S–ABC. Hence, the three pyramids into which the prism ABC–ESD is divided are equivalent and, therefore, the pyramid S–ABC is equivalent to one third of the prism. But the volume of the prism is equal to $B \times H$; therefore volume of pyramid is given by the formula $V = \frac{1}{3} B \times H$.

As cones can be regarded as the limits of pyramids when the number of sides is indefinitely increased, the lateral area of a cone equals one-half the slant height by the circumference of the base, and the volume equals one-third the altitude by the area of the base. The formulas are $S = \pi rl$, and $V = \frac{1}{3}\pi r^2 h$, where S = lateral surface, V = volume, l = slant height, r = radius of base and h = altitude.

7. The Prismoidal Formula.—A prismoid has as bases two polygons in parallel planes, as shown in Fig. 93. Its lateral faces are either triangles or trapezoids whose bases are sides of one polygon and the opposite vertex or base is common to the other polygon.

Theorem: *The volume of a prismoid is equal to the product of one-sixth of its altitude into the sum of its bases and four times its mid-section.*

Let V denote the volume, B and b the bases, M the mid-section, and H the altitude, of a given prismoid. Then the formula is

$$V = \frac{1}{6} H(B + b + 4M).$$

This formula can be used to find the volume of all the solids we have used in geometry. It is true not only for the prism, pyramid, and frustrum of a pyramid, but also for the cone and its frustrum, the cylinder and the sphere. The proof is indicated by the figure.

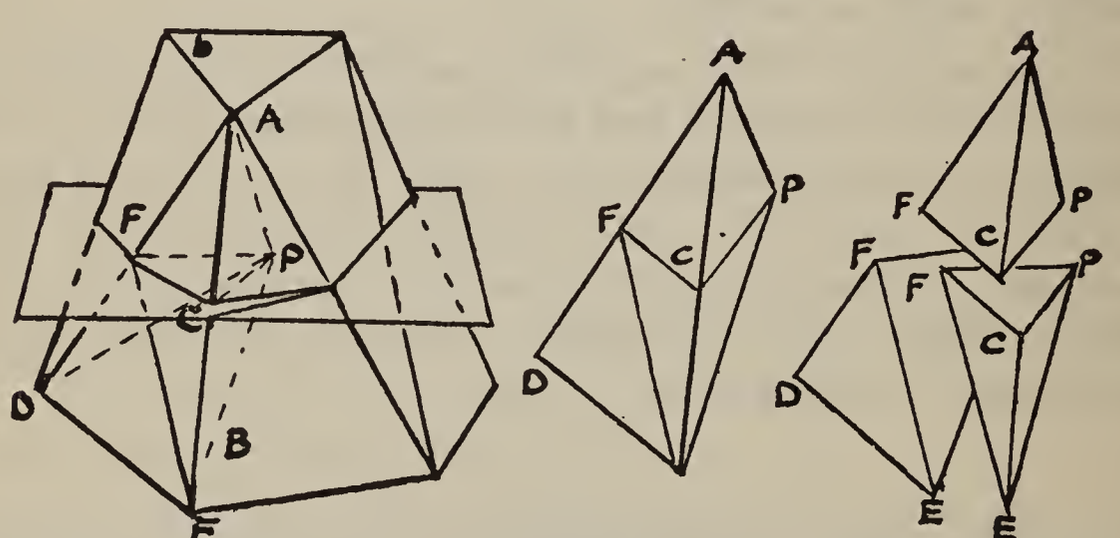

Fig. 93.

8. Similar Solids.—We learned in Plane Geometry that the triangle was rigid and that all polygons could be decomposed into triangles. In Solid Geometry the tetrahedron is likewise rigid and all polyhedra can be decomposed into tetrahedrons. *Tetra* means four and tetrahedrons have four faces, all of which, we will

understand, are triangles. The tetrahedron is also called a *triangular pyramid.* Any two opposite edges of a tetrahedron may be regarded as diagonals and three different quadrilaterals in space considered.

Theorem: *The volumes of two tetrahedrons, having a trihedral angle of the one equal to a trihedral angle of the other, are to each other as the products of the three edges of these trihedral angles.*

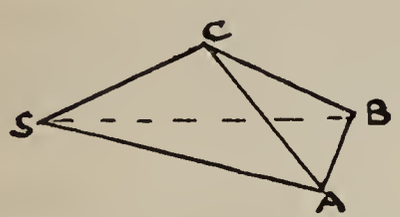

Fig. 94.

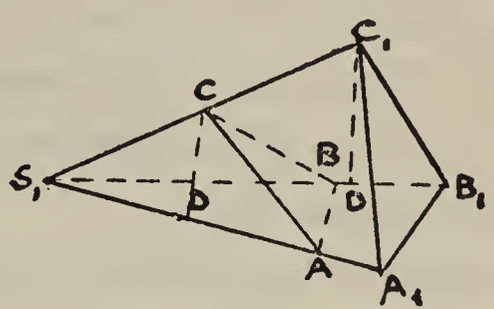

Fig. 95.

Let V and V_1 denote the volumes of the two triangular pyramids S–ABC and S_1–$A_1B_1C_1$, Figs. 94 and 95, having the trihedral angles S and S_1 equal.

To prove that $\frac{V}{V_1} = \frac{SA \times SB \times SC}{S_1A_1 \times S_1B_1 \times S_1C_1}$, place the pyramid S–ABC upon S_1–$A_1B_1C_1$ so that the trihedral angle S shall coincide with S_1. Draw CD and C_1D perpendicular to the plane $S_1A_1B_1$ and let their plane intersect $S_1A_1B_1$ in S_1DD.

The faces S_1AB and $S_1A_1B_1$ may be taken as the bases, and CD, C_1D as the altitudes, of the triangular pyramids C–S_1AB and C_1–$S_1A_1B_1$, respectively.

$$\text{Then } \frac{V}{V_1} = \frac{S_1AB \times CD}{S_1A_1B_1 \times C_1D} = \frac{S_1AB}{S_1A_1B_1} \times \frac{CD}{C_1D}$$

$$\text{But } \frac{S_1AB}{S_1A_1B_1} = \frac{S_1A \times S_1B}{S_1A_1 \times S_1B_1} \text{ as triangles that have equal}$$

angles are to each other as the sides about the equal angles.

$$\text{Therefore } \frac{V}{V_1} = \frac{S_1A}{S_1A_1} \times \frac{S_1B}{S_1B_1} \times \frac{S_1C}{S_1C_1} = \frac{S_1A \times S_1B \times S_1C}{S_1A_1 \times S_1B_1 \times S_1C_1}$$

If the tetrahedrons are similar

$$\frac{S_1A}{S_1A_1} = \frac{SB}{S_1B_1} = \frac{SC}{S_1C_1}, \text{ whence } \frac{V}{V_1} = \frac{SA^3}{S_1A_1{}^3}$$

And as similar polyhedrons can be decomposed into similar tetrahedrons, this enables us to say that *similar solids are to each other as the cubes of their like dimensions.*

If *homologous*, that is, *like placed* lines of two similar polyhedrons are *six* and *seven* inches respectively, then the volumes are as $\frac{6\times6\times6}{7\times7\times7}=\frac{216}{343}$, which is in accordance with the teaching of arithmetic that if the edge of one cube is three times the edge of another its volume is $3 \times 3 \times 3 = 27$ times as great.

Two important theorems about similar figures are:

1. *Similar plane figures are to each other as the squares of homologous lines.*

2. *Similar solid figures are to each other as the cubes of homologous lines.*

Bearing these theorems in mind we will consider two similar tetrahedrons, where planes trisect three edges, as in Fig. 96.

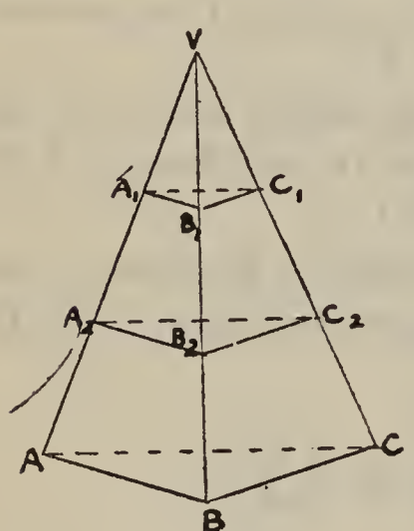

Fig. 96.

Evidently the pyramids V–$A_1B_1C_1$, V–$A_2B_2C_2$, V–ABC are similar and are to each other as 1^3 to 2^3 to 3^3, that is, as $1 : 8 : 27$. Thus V–$A_1B_1C_1$ is $\frac{1}{27}$ of V–ABC, and V–$A_2B_2C_2$ is $\frac{8}{27}$ of V–ABC. By subtraction it is readily seen that the frustrum $(A_1B_1C_1, A_2B_2C_2)$ is $\frac{8}{27}-\frac{1}{27}=\frac{7}{27}$ of the whole figure $V - ABC$, and that $(ABC - A_2B_2C_2)$ is $\frac{27}{27}-\frac{8}{27}=\frac{19}{27}$ of the whole figure. Thus by dividing the edges into three equal parts we divided this volume into parts which were as 1 to 7 to 19.

As the triangles $A_1B_1C_1$, $A_2B_2C_2$, ABC are similar they are as 1^2 to 2^2 to 3^2 or as $1 : 4 : 9$, which reminds us of such expressions as the physical law that light varies in intensity as the *square* of its distance from the source. It is evident that these principles hold when the pyramids have bases other than triangles.

9. Spherical Geometry.—A sphere is a solid bounded by a surface all points of which are equally distant from a point within called the center. This solid may be generated by the revolution of a semicircle, as *ACB* in Fig. 97, about its diameter *AB* as an axis.

The point *C* describes a *great circle* of the sphere, that is, a circle whose plane passes thru the center of the sphere; all other points on the generating semicircle describe *small* circles. At *B* and *A* are poles of all these circles. In Plane Geometry we defined a straight line as the shortest distance between two points; in spherical geometry the arc of a great circle is the shortest distance between two points. This is why we sail quite a distance north in going to Europe. Evidently every section of a sphere is a circle and all points of a circle are equidistant from each pole. A great circle is at a quadrant's distance from its poles.

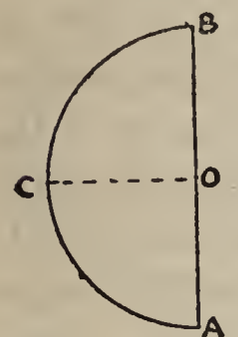

Fig. 97.

The *radius* of a sphere is a straight line from the center to the periphery or surface of the sphere.

The *diameter* of a sphere is a straight line thru its center and terminated by two points of its surface.

The *tangent* of a sphere is a straight line which touches the sphere at only one point of its surface.

Given a material sphere to find its diameter.

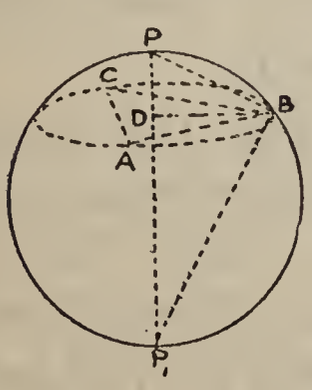

Fig. 98.

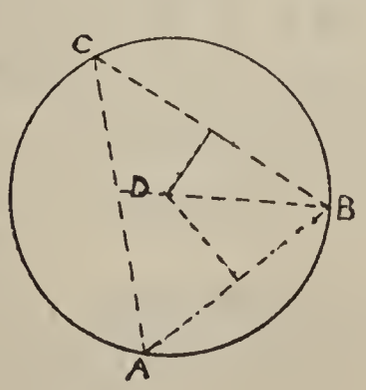

Fig. 99.

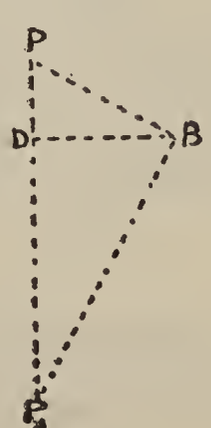

Fig. 100.

Suppose we wish to find the diameter of a ball represented by Fig. 98. With P as a pole and with a pair of curved compasses opened any distance PB, we construct a circle and mark any three points ABC on it. Measuring the three distances, AB, BC, CA with the compasses, we next construct on a piece of paper the triangle ABC, and by bisecting its sides, drawing perpendiculars and circumscribing the circle, we have Fig. 99, in which DB is the radius of the circle we drew. Now passing to Fig. 100 we first draw the line DB equal to this radius then a line PD, perpendicular to DB. With a radius equal to BP, the first opening of the compasses, and with a center at B, we construct an arc cutting the perpendicular at P. If now we draw BP_1 perpendicular to BP, PP_1 will be the diameter of our ball or sphere, for the angle PBP_1 being a right angle is inscribed in the generating semicircle. By similar process it can be shown that a sphere may be inscribed in any tetrahedron and another sphere about it.

10. Figures on the Surface of a Sphere.—The only figures on a sphere that are considered in elementary geometry are those whose sides are great-circle arcs. The angle between any two arcs is defined as the angle formed by the tangents C_1A, B_1A, Fig. 101, to the arcs at this point of meeting. It is also the plane angle COB of the dihedral angle of the planes of these circles, hence arc CB (often written $\overset{\frown}{CB}$) at a quadrant's distance from A measures the angle A.

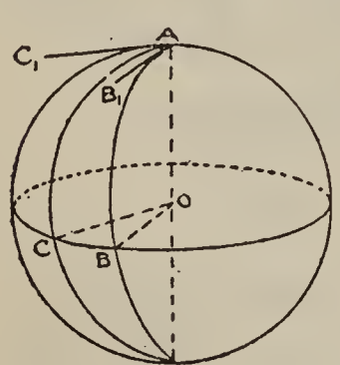

Fig. 101.

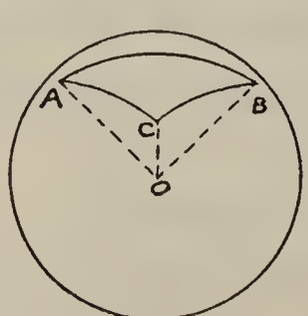

Fig. 102.

The theorems about equality of triangles and their angles correspond to those of plane geometry.

In Fig. 102 is shown a spherical pyramid $O - ABC$, which illustrates the method of proof of many truths in spherical geometry connecting a spherical polygon with the polyhedral angle at the center.

We see that the angle AOC has the same number of degrees as the arc AC, etc. Since we know that angle $AOC + COB$ is greater than AOB, so $\overset{\frown}{AC} + \overset{\frown}{CB}$ is greater than $\overset{\frown}{AB}$. Also, since $AOC + COB + AOB$ is less than 360°, $\overset{\frown}{AC} + \overset{\frown}{CB} + \overset{\frown}{BA}$ is less than 360°, and the sides of a spherical polygon contain less than 360°.

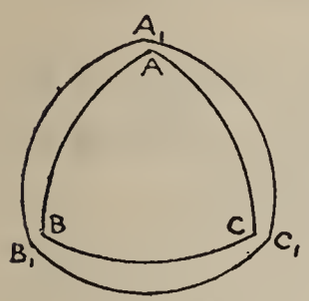

Fig. 103.

If from the vertices of a spherical triangle as poles, arcs of great circles are described, eight other spherical triangles will be formed. The one whose vertices lie nearest to those of the original triangle is called the *polar triangle*. From the manner of construction either $A_1B_1C_1$ or ABC in Fig. 103 is the polar triangle of the other.

11. Angles of Spherical Triangles.—The next of our theorems enables us to say that if the angles of a spherical triangle are equal the sides of its polar triangles are also equal and reciprocal.

Theorem: *In two polar triangles, each angle of one has the same measure as the supplement of that side of the other of which its vertex it is the pole.*

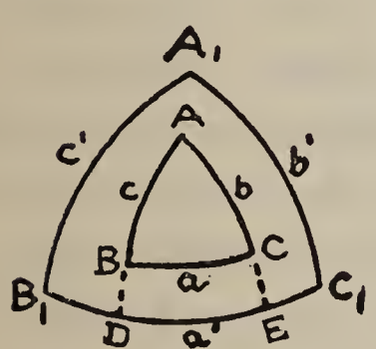

Fig. 104.

Triangles ABC and $A_1B_1C_1$ of Fig. 104 are polar, the point A being the pole of the arc B_1C_1, etc. Let a, a_1, etc., be the measures in degrees of the arcs BC, B_1C_1, etc., respectively, and let A, A_1, etc., be the measures in degrees of the angles A, A_1, etc. Then,

$$A = 180 - a_1; \quad B = 180 - b_1; \quad C = 180 - c_1.$$

$$A_1 = 180 - a; \quad B_1 = 180 - b; \quad C_1 = 180 - c.$$

In proof of this, extend the arcs AB and AC to meet the arc B_1C_1 at D and E, respectively. Then, since B_1 is the pole of the arc AE, the arc $B_1E = 90$.

Similarly, the arc $C_1D = 90°$; therefore $\overset{\frown}{B_1E} + \overset{\frown}{C_1D} = 180°$, and $\overset{\frown}{B_1D} + \overset{\frown}{DE} + \overset{\frown}{DC_1} = 180°$, or $\overset{\frown}{DE} + \overset{\frown}{B_1C_1} = 180°$. But $\overset{\frown}{DE}$ is the measure of the angle A; therefore,

$$A + a_1 = 180, \text{ or}$$
$$A = 180 - a_1$$

Similarly for each of the other angles of the triangle.

Theorem: *The sum of the angles of a spherical triangle is greater than* 180° *and less then* 540°.

Proof: Let the triangle $A_1B_1C_1$ be the polar triangle of the spherical triangle ABC, Figs. 104 and 105, A being the pole of the arc B_1C_1, B of the arc A_1B_1 and C of the arc A_1B_1.

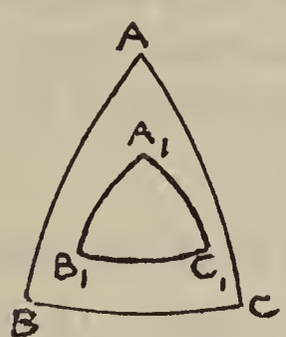

Fig. 105.

Then $A = 180° - a_1$, $B = 180° - b_1$, and $C = 180 - c_1$.

Also $A + B + C = 540^c - (a_1 + b_1 + c_1)$.

Therefore $A + B + C$ is less than 540°.

But $a_1 + b_1 + c_1$ is less than 360°.

Therefore $A + B + C$ is greater than 180°.

Evidently, *the sum of the angles of a spherical polygon of n sides is greater than* $(n - 2) \times 180°$.

The spherical excess E of a polygon is the difference between the sum of its angles and $(n - 2) \times 180°$. (In a triangle $n = 3$ and $n - 2 = 1$).

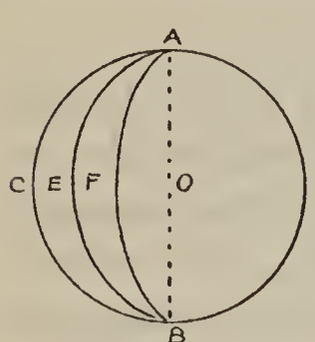

Fig. 106.

A *lune* is the portion of a spherical surface bounded by two semi-great circles as *AEFB* in Fig. 106. When an orange is peeled the spherical surface of one section of it is a lune, the section itself forming a spherical wedge. Both a lune and the corresponding spherical wedge are evidently the same part of the surface and volume of a sphere as the angle of the lune is of 360°. We say, therefore, that a lune whose angle is one degree contains two spherical degrees, there being 720 spherical degrees on a sphere.

Theorem: *Two symmetrical spherical triangles are equivalent.*

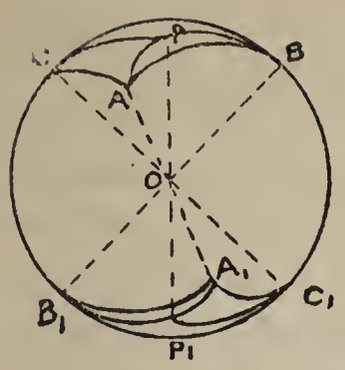

Fig. 107.

Let ABC, $A_1B_1C_1$ of Fig. 107 be two symmetrical spherical triangles with their homologous vertices opposite each to each. To prove that the triangles ABC, $A_1B_1C_1$, are equivalent, let P be the pole of a small circle passing thru the points A, B, C, and let POP_1 be a diameter. Then draw the great circle arcs $\overset{\frown}{PA}$, $\overset{\frown}{PB}$, $\overset{\frown}{PC}$, $\overset{\frown}{P_1A_1}$, $\overset{\frown}{P_1B_1}$, $\overset{\frown}{P_1C_1}$. In that case $\overset{\frown}{PA} = \overset{\frown}{PB} = \overset{\frown}{PC}$ (Polar distances). Now $\overset{\frown}{P_1A_1} = \overset{\frown}{PA}$, $\overset{\frown}{P_1B_1} = \overset{\frown}{PB}$, $\overset{\frown}{P_1C_1} = \overset{\frown}{PC}$, since they measure vertical central angles.

Therefore, $P_1A_1 = P_1B_1 = P_1C_1$ and the two symmetrical triangles PAC and $P_1A_1C_1$ are isosceles. Accordingly the triangle PAC = the triangle $P_1A_1C_1$ as they are equal in like order as well as in reverse.

Similarly, the triangle PAB = the triangle $P_1A_1B_1$, and the triangle PBC = the triangle $P_1B_1C_1$. Now the triangle ABC is equivalent to the triangle PAC plus the triangle PAB plus the triangle PBC, and the triangle $A_1B_1C_1$ is equivalent to the triangle $P_1A_1C_1$ plus the triangle $P_1A_1B_1$ plus the triangle $P_1B_1C_1$.

Therefore the triangle ABC is equivalent to the triangle $A_1B_1C_1$.

If the pole P should fall without the triangle ABC, then P_1 would fall without the triangle $A_1B_1C_1$ and each triangle would be equivalent to the sum of two symmetrical isosceles triangles diminished by the third; so that the result would be the same as before. The discovery of a third case is left to the reader

12. Areas of Spherical Triangles.—There is a theorem which says that *the area of a spherical triangle, expressed in spherical degrees, is numerically equal to the spherical excess of the triangle.*

Let A, B, C, Fig. 108, denote the values of the angles of the spherical triangle ABC, and E the spherical excess. To prove that the number of spherical degrees in the angle ABC equals E, produce the sides of the triangle ABC to complete circles. These circles divide the surface of the sphere into eight spherical triangles, of which any four having a common vertex as A, form the surface of a hemisphere.

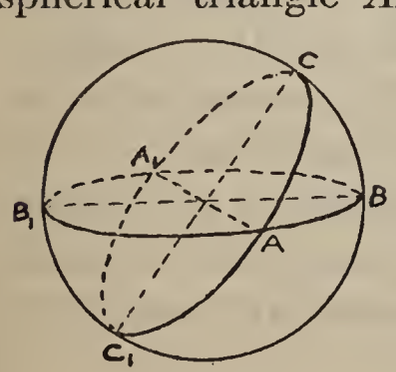

Fig. 108

The triangles A_1BC, AB_1C_1 are sym-

metrical and equivalent, and the triangle ABC plus the triangle A_1BC is equivalent to lune ABA_1C. Put the triangle AB_1C_1 for its equivalent, the triangle A_1BC. Then the triangle ABC plus the triangle AB_1C_1 is equivalent to lune ABA_1C; also the triangle ABC plus the triangle AB_1C is equivalent to lune BAB_1C, and the triangle ABC plus the triangle ABC_1 is equivalent to lune CAC_1B. Add and observe that in spherical degrees the triangle ABC plus the triangle AB_1C_1 plus the triangle AB_1C plus the triangle $ABC_1 = 360°$ (half of 720°) and

$$ABA_1C + BAB_1C + CAC_1B = 2(A + B + C).$$

Then twice the triangle $ABC + 360° = 2(A + B + C)$.

Therefore the triangle $ABC = A + B + C - 180° = E$.

A triangle whose angles are 100°, 110°, 120° has a spherical excess of $(100° + 110° + 120° - 180°) = 150$ spherical degrees and contains $\frac{150}{720} = \frac{5}{24}$ of the surface of the sphere.

Theorem: *The area of the surface generated by the revolution of a straight line about a straight line in its plane, not parallel to and not intersecting it, as an axis, is equal to its projection on the axis, multiplied by the circumference of a circle, whose radius is the perpendicular erected at the mid-point of the line and terminating in the axis.*

The straight line AB in Fig. 109 is revolved about the straight line M in its plane, not perpendicular to it and not intersecting it, as an axis. The lines AC and BD are perpendicular to OM, and EO is the perpendicular erected at the mid-point of AB terminating in OM. Then area AB (that is, the area of the surface generated by AB) $= CD \times 2\pi EO$.

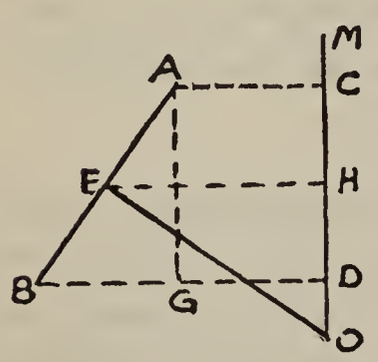

Fig. 109.

Proof: Draw line AG perpendicular to BD, and line EH perpendicular to CD The surface generated by AB is the lateral surface of a frustrum of a cone of revolution, whose bases are generated by AC and BD.

Therefore area $AB = AB \times 2\pi EH$. Triangles ABG and EOH are similar. Therefore, we have

$\frac{AB}{AG} = \frac{EO}{EH}$ whence $AB \times EH = AG \times EO = CD \times EO$.

Substituting in first equation, area $AB = CD \times 2\pi EO$.

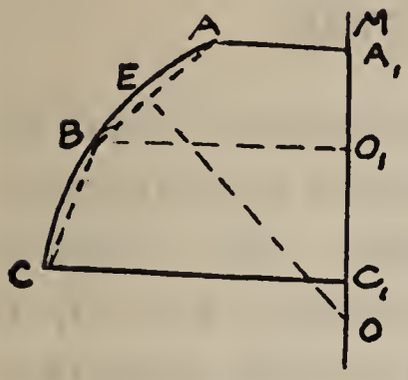

Fig. 110.

As more chords AB, Fig. 110, are inscribed in the arc AC, the area generated by them will approach as a limit the area generated by the arc. This area is called a *zone*, its area is $A_1C_1 \times 2\pi R$ where EO becomes the radius. If the arc AC were a semicircle it would be the generating semicircle and the area of the sphere would be $2R \times 2\pi R = 4\pi R^2$. The solid, of which a zone is the spherical surface, is called a *spherical segment* when the bounding planes are parallel. There is another spherical solid of which the zone is the spherical surface and it is generated by a sector of a circle revolving about a diameter. There are three in Fig. 111.

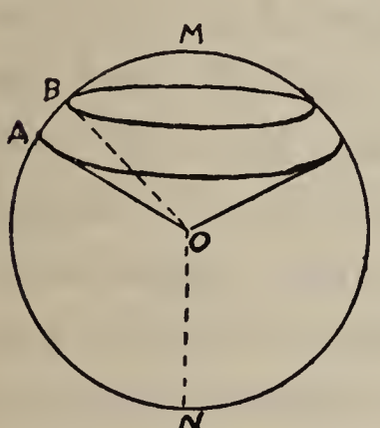

Fig. 111.

We learned that the volume of a pyramid was equal to one-third the product of its base by its altitude. The sphere could be thought of as composed of innumerable spherical pyramids with their vertices at the center, hence its volume is $\frac{1}{3} R \times 4\pi R^2 = \frac{4}{3} \pi R^3$.

REVIEW.

1. What is a dihedral angle? A polyhedral angle? Give a familiar example of the latter.

2. Define prism, a right prism, a parallelopiped.

3. What is the rule for finding the lateral surface of a pyramid and a cone? Give the formula for their volume.

4. How are similar solids related to each other?

5. Define sphere, radius of a sphere, diameter and tangent.

6. How does a spherical triangle differ from a plane triangle?

CHAPTER VII

ALGEBRA—NUMBER SYSTEM

1. New Kinds of Number.—The first numbers used by man in the long course of his intellectual development were those of elementary arithmetic, the *positive integral numbers* as they are called, namely 1, 2, 3, 4, 5, etc. We know from arithmetic that the operation of addition performed on these numbers gives no new numbers as a result, only other numbers of the same set. Likewise, as multiplication is only abbreviated addition, this operation could give only numbers of the same set. The inquiring mind which found that $3+2$ produced 5 was also led to see the fact that $5-3$ produced 2 and $5-2$ produced 3 and that no new kind of number was thus obtained.

But when some inquisitive mind wanted to perform such an original operation as $3-5$, which had no meaning in arithmetic, he found out that he was attempting the opposite of the operation $5-3=2$ and concluded that his answer should be the opposite of 2. Now to him the opposite of having two horses was to owe two horses, and as his first idea of opposition was addition and subtraction, he used the signs of these operations (+) and (−) in the new sense of quality. Thus man came to invent *negative numbers*, which are the opposites of positive numbers, and the number system became

$$\ldots -8, -7, -6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, +6, +7, +8 \ldots$$

where every number has its opposite. Thus $+2$ and -2 are opposites, also -5 and $+5$. A man who

owes one dollar needs to earn two dollars before he can claim he has a dollar. Hence $+1$ is two more than -1, so we put 0 in between these numbers, and say 1 more than -1 is 0, and 1 more than 0 is $+1$, and the numbers in our revised number system, as in the old one, increase by 1 from left to right. The 0 (read zero) is its own opposite, $+0$ and -0 are the same, and no sign need be written.

2. The Quality of Positive and Negative.—These numbers will add and subtract either way and still give numbers of their set. If the temperature on a winter's day was 4° below zero in the morning, that is $-4°$, and 28° above zero, that is $+28°$, at noon, the rise in temperature is 32°. If the temperature falls 30° by midnight, the temperature then will be $-2°$.

By looking at the succession of numbers in our revised system, we can readily see that -5 from 4 is 9, for by counting the numbers after -5 until we come to 4 we find there are nine of them. Similarly, to find what $3-8$ equals we will have to count back 8 steps from 3, which brings us to -5. Likewise $-2-6$ will be 6 steps backward from -2, which will bring us to -8. Notice that $+$ and $-$ have now two meanings, so to speak, the $-$ before 2 in this instance meaning quality, and the $-$ before 6 indicating the operation of subtraction. It will be seen, therefore, that these negative numbers merely extend our number system the other way. *The idea is entirely one of opposition.* If A and B have had business dealings, what A would call positive in their relationships, B would call negative from his point of view. What B would owe A, A would count in his assets and B in his liabilities.

The idea of $+$ and $-$ is found in temperature, in electricity, in measurement above and below sea level

and wherever the idea of *opposition* is to be expressed. We have positive and negative latitude; positive, north of the equator, and negative south of the equator. Had civilization and the science of navigation developed south of the equator, southern latitude would no doubt have been positive and northern negative, since it is a matter of habit to consider the original way $+$ and the other way $-$. When there is no opposition there is, of course, no sign. When opposition exists we need both signs.

3. The Principle of Permanence.—The positive and negative whole numbers constitute a number set in which the operations of addition and subtraction produce no new number. The negative numbers were added to our number system in order to preserve this uniformity. This attempt to preserve uniformity is called the *Principle of Permanence*, or the *Principle of No Exception*. We shall see that multiplication, since it is merely abbreviated addition, does not produce any new numbers when performed on numbers of our set.

If A owes \$2 for one day's board and remains another day under these conditions, he will owe \$2 and then \$2 more, owing altogether \$4. If he stays one week, he will be owing 7 times \$2 or \$14, or in symbols, $7\times-2=-14$. And as multiplication is commutative, we could also say $-2\times7=-14$. It is probably easier to see that -2×7 is the opposite of 2×7, and since $2\times7=14$ it follows that $-2\times7=-14$.

In the army "right about face" turns a column of soldiers in the opposite direction from that in which they were standing or marching. Marching in the new direction would be negative if we considered the first

direction positive. As we have seen, − before either number of a product gives opposition and we would expect $-3\times5=-15$ and $3\times-5=-15$. But just as two successive "right about face" orders would find the column marching in the original direction, so when each of the two numbers of a product is *minus* (−) the result is *plus* (+). An odd number of − signs in continued multiplication or division will produce minus, just as an odd number of right-about-face orders will find the column marching in a direction contrary to the first direction; while an even number of minus signs, like an even number of "right about face" orders, will be the equivalent of the command "As you were." We can further explain this point by showing that $-2\times-3=6$ since -2×-3 is the opposite of either 2×-3 or -2×3, each of which produce -6 and the opposite of -6 is 6.

It is to be noted that with this interpretation of signs multiplication has added no new numbers to our number system of positive and negative whole numbers together with 0.

4. Fractions a New Kind of Number.—As subtraction was the inverse of addition, so is division the inverse of multiplication. We learned in Arithmetic that since $3\times4=12$, then $12\div4=3$ and $12\div3=4$; but there was no meaning to $5\div7$ or $8\div3$, because no number in our table multiplied by 7 equals 5, neither was $3\times$ any number equal to 8. We had to introduce into our set the idea of fractional numbers in order to make possible these very desirable operations.

There are negative fractional numbers as well as positive fractional numbers. Experimentation will show that the operations of addition, subtraction,

multiplication and division will produce no necessity for numbers other than positive and negative whole numbers and fractions.

5. Power, Root and Logarithm.—When a number, say 2, is to be multiplied by itself a number of times, we do not write $2\times2\times2=8$, but economize time and space by writing $2^3=8$. This is read "two to the third power" equals eight or "two cubed" equals eight. The small 3 placed to the right and partly above, the 2 as shown, is called an *exponent* and indicates that 2 is to be used three times as a factor. Since 2 is one of the 3 equal factors of 8, we call 2 the cube root of 8, written $\sqrt[3]{8}=2$, where the term *cube root* means one of the *three* equal factors. The term *square root* means one of *two* equal factors. Thus $\sqrt{9}=3$, reads "square root of 9 equals 3." *Fourth root* is one of the *four* equal factors. The cube root of -8 written $\sqrt[3]{-8}=-2$, since $-2\times-2\times-2=-8$. The *fifth root* of $-243=-3$, written $\sqrt[5]{-243}=-3$. There is also in the case $2^3=8$ another inverse. We say, 3 is the *logarithm* of 8 to *base* 2, written $3=\log_2 8$. That is, 8 can be regarded as a power of 2. Looked at from this point of view, exponents may be logarithms. We might also say that 4 was the logarithm of 16 to the base 2 since $2^4=16$. We should notice that these two inverses are *distinct*, which was not the case in subtraction and division.

6. Irrationals and Imaginary Numbers.—The expression $\sqrt{2}$ has as yet no definite meaning for us, for we can find no number in our system of positive, negative and fractional numbers which when multiplied by itself will exactly produce 2. Such a number must, however, be somewhere in between 1 and 2. A series

of multiplication tests beginning with 1.4×1.4 and 1.5×1.5 will result in our finding the square root of 2 lying somewhere between 1.4142 and 1.4143. The last two of these tests are here given in the approximate form.

1.4142	1.4143
1.4142	1.4143
1.4142	1.4143
5657	5657
141	141
56	56
3	4
1.9999	2.0001

We could find more places of decimals by the same process. Also $\sqrt{3}$, $\sqrt{5}$, etc., could be found by this same method, which is still much used by older mechanics.

It will be well to memorize the fact that the square root of 2 is 1.414 approximately and that the square root of 3 is 1.732 approximately, since these figures occur very frequently in both mathematics and physics. Other such numbers of a similar nature are $\sqrt[3]{9}$, $\sqrt[5]{33}$, etc. None of these numbers is found in our number system of positive and negative whole and fractional numbers. These new numbers we call *irrational*, but sometimes are referred to as *surds* and *radicals*.

We have already noticed that $-3\times-3=+9$, also that $+3\times+3=+9$. That is, the square of any positive or negative number always is a positive number. This being so, what then is the meaning of the square root of -4, written $\sqrt{-4}$? This, evidently, has no meaning until we say that all numbers whose square is a negative number are *imaginary numbers*.

7. Transcendental Numbers.—We learned in Section 5 of this chapter that $8=2^3$ and found two inverses, $2=\sqrt[3]{8}$ and $3=\log_2 8$. Now since $2^4=16$, we likewise say $4=\log_2 16$. Now what do $\log_2 9$, $\log_2 10$, $\log_2 11$, $\log_2 12$, $\log_2 13$, $\log_2 14$, $\log_2 15$ equal? Evidently they are all different numbers, somewhere between 3 and 4. They will be found to differ from any of the numbers we have discovered before, and in fact are one type of a class of numbers called *transcendental.* We shall find these numbers very useful in computation and shall have occasion to use them frequently. Other types of transcendental numbers occur in algebra, geometry, trigonometry and calculus.

Let us call to mind that there were *seven fundamental operations,* namely, addition and its inverse subtraction, multiplication and its inverse division, raising to a power and its two inverses root and logarithm. The first inverse, subtraction, gave us negative numbers; the second inverse, division, gave us fractions; the third inverse, root, gave us both irrationals and imaginaries; the fourth inverse gave us logarithms. Algebra treats of a number system composed of all these types of numbers. *It excludes division by zero* and gives no meaning to the logarithm of any but *positive* numbers. We are now in a position to obtain a comfortable acquaintance with these numbers, feeling absolutely certain that the *principle of permanence* will guide us safely.

8. Constant and Variable Numbers.—Viewing numbers from another standpoint, we may divide them into *constants* and *variables.* Constant numbers never change their value. There are very few such numbers. The time between two fixed events is a constant.

The ratio of the circumference to the diameter of the circle is a constant whose exact value, however, has never been found, tho it has been expressed correctly to 707 decimal places.

Such extended computation is, however, a curiosity rather than a useful result. The ratio of circumference to diameter is to different degrees of approximation 3, $\frac{22}{7}$, $\frac{355}{113}$, or in decimals, 3.14159. The first value is mentioned twice in the Bible, the second and third were known to the Chinese and Babylonians 4000 years ago. Another member of the family of constants is the base of the natural system of logarithms, which is 2.718..... The exact value of these two numbers can never be determined. They are types of transcendental numbers.

A quantity which is subject to change is called a *variable*. There are many such quantities. One of them is *time*. The time following any event increases continuously and eternally. Such a variable is said to be *infinite*. We speak of its quality, not its quantity. When we say that the time following your first opening of this book is infinite we mean that it has the quality of continuously increasing for ever and ever. Infinity is not an immense number or size, but the quality of growing larger and larger. It is a variable that continuously increases. The opposite of a variable of that kind is an *infinitesimal*. The remainder of one's life is an infinitesimal because it continuously decreases. Other variables are the speed of a train, the steam-pressure in a boiler, the velocity of the wind, the temperature of the air and the height of the barometer. The value of a variable at any instant is a constant.

9. Functions and Limit of a Variable.—Some variables depend upon other variables. The time it takes

to walk to the railroad station might be said to be a variable depending upon one's speed. Often the speed at which one runs to catch a train depends upon the time allowed oneself. In the first instance, the time is said to be a *function* of the speed; in the second, the speed is a function of the time. In some cases, one variable is a function of several variables. Our ability to master mathematics is probably a function of our power of attention, our persistence, our innate ability and many other qualities which it would be well for us to examine carefully.

In football there is a rule which, when broken, penalizes a team one-half the distance between the place of the transgression and the offending team's own goal. If this penalty were repeatedly applied, could the opposing team ever score by a penalty? Would that team not always be just as far from the goal as the latest penalty had carried it? The sum of the successive values of this variable, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, would never quite equal 1, but the difference between this sum and 1 *would become and remain less than any assigned value, no matter how small.* This difference between the sum and 1 is evidently an infinitesimal. When a variable differs from a constant by an infinitesimal, we call the constant *the limit of the variable.*

REVIEW.

1. What is meant by "negative numbers"? Give a brief explanation of the quality of positive and negative.

2. Why are fractions necessary?

3. What is an "exponent" and what does it indicate? State the difference between "root" and "power."

4. Of what types of numbers does algebra treat?

5. Give some account of "constant" and "variable" numbers.

6. Explain the meaning of "infinite."

CHAPTER VIII

SIMPLE EQUATIONS AND FUNDAMENTAL OPERATIONS

1. The Equation.—In algebra variables are often represented by x, or some other letter. Of two variables one could be represented by x and the other by y. If these two variables were related so that 3 times one of the variables added to 4 times the other variable equaled 7, we would write the expression in algebraic shorthand, $3x+4y=7$. It is very evident that when x equals 1, y must also equal 1, because $3x+4y=7$ becomes the *identity* $3+4=7$. The expression $3x+4y=7$ is called *an equation*. The value we have given to x determines the value we must give to y. Thus if x equals 1, as we have seen, y too must equal 1. Now if x were equal to -5, the proper value for y would be -2.

The following is a table of a few of the values which x and y may have simultaneously in the equation $3x+4y=7$. We could never write all the values.

If $x=1$, $y=1$
If $x=2$, $y=\frac{1}{4}$
If $x=3$, $y=-\frac{1}{2}$
If $x=4$, $y=-\frac{5}{4}$
If $x=5$, $y=-2$
If $x=0$, $y=\frac{7}{4}$
If $x=-1$, $y=\frac{5}{2}$
If $x=-2$, $y=\frac{13}{4}$
If $x=-3$, $y=4$
If $x=-4$, $y=\frac{19}{4}$

If $x=-1$, $y=1$
If $x=-\frac{1}{3}$, $y=2$
If $x=-\frac{5}{3}$, $y=3$
If $x=-3$, $y=4$
If $x=-\frac{13}{3}$, $y=5$
If $x=\frac{7}{3}$, $y=0$
If $x=\frac{4}{3}$, $y=-1$
If $x=5$, $y=-2$
If $x=\frac{19}{3}$, $y=-3$
If $x=\frac{23}{3}$, $y=-4$

2. Limitations of Variables in an Equation.—It will be noticed that 3 times any x in the table plus 4 times its y always equals 7. We have called $3x+4y=7$ an equation. It is very much like a pair of scales. The right-hand member always has to balance with the left-hand member. We could have found these numbers either by giving value to x and "solving" for y or *vice versa*. Thus, if x equals 5, $3x$ will equal 15 and our equation becomes $15+4y=7$. Now if we subtract 15 from each side of this balance, the two sides will still be equal and we will have $4y=-8$, and $y=-2$, since y is clearly $\frac{1}{4}$ of $4y$. Likewise if x is equal to 3, then $3x=9$ and we have, "substituting," $9+4y=7$. Subtracting 9 from each side of the equation leaves us with $4y=-2$, and $y=-\frac{2}{4}$ or $-\frac{1}{2}$. In the same manner, if y equals 0, then $3x+4y=7$ becomes $3x=7$, whence $x=\frac{7}{3}$. It will be well to verify all the figures for simultaneous values of x and y and to obtain others.

Now x could have any positive or negative value, fractional or integral, and y would have to be determined to fit or we could give values to y and thus determine x. If x were not an exact number, neither would y be.

The idea of equation is fundamental in algebra, and letters like x, y, z, etc., are used to represent variables which may have any value at a particular instant. These letters may also represent numbers whose value we wish to determine and since we do not know the exact numerical value for these numbers we often call them *unknown*. In any of these meanings they represent any of a succession of particular numbers until we exactly determine what value we can assign.

3. The Use of the Equation.—Let us suppose that a certain number added to itself is equal to 250. What is the number?

We can represent this undetermined number by x. Now x added to x is certainly $2x$; therefore in algebraic shorthand we can say that $2x=250$. Whence it follows that x equals 125. Note that the algebraic notation is not only shorter but more suggestive of the calculation involved.

Again, suppose the expenses of a factory doubled each year for 3 years and that the third year they were \$13,800. We want to know what were the expenses for each of the other years.

If we knew what the expenses were the first year, the problem would be practically solved. Therefore we will let x represent the *number* of dollars of expenses for the first year. Since the expenses doubled, they could be represented by $2x$ in the second year and since they again doubled they could be represented by $4x$ in the third year. But the problem says that the expenses for the third year were \$13,800, therefore

$4x=\$13{,}800$, from which it follows that
$x=\ \$3{,}450$ (the expenses of the first year)
and that $2x=\ \$6{,}900$ (the expenses for the second year.)

Suppose a man has two daughters and one son. He wishes to divide \$6000 among them so as to give the elder daughter twice as much as the younger, and the son as much as both the daughters. He wants to know how much must be given to each.

If we knew what he must give to the younger daughter we would also know what he must give to the elder daughter, and accordingly we could easily find out what he must give to the son. Let x equal the *number* of dollars he is to give the younger daughter. Then certainly $2x$ is the number of dollars he must give to the elder daughter and the sum of these, $3x$, is therefore the number

of dollars the son will receive. These three numbers, x, $2x$ and $3x$ added together make $6x$. Evidently, then,

$6x = \$6000$
$x = 1000$ (number of dollars youngest daughter will receive.)
$2x = 2000$ (number of dollars elder daughter will receive.)
$3x = 3000$ (number of dollars son will receive.)

4. Examples in Equations.—Suppose A can do a piece of work in 8 days, B in 10 days, and C in 24 days. In what time will they finish it if working together?

Let x represent the number of days in which they can finish it, all working together. In 1 day they can therefore do $\frac{1}{x}$ of the work, to which A will contribute $\frac{1}{8}$ of the whole work, B $\frac{1}{10}$ and C $\frac{1}{24}$. Put in the form of an equation, this will be:

$$\frac{1}{x} = \frac{1}{8} + \frac{1}{10} + \frac{1}{24} \quad \text{whence}$$

$$\frac{1}{x} = \frac{15 + 12 + 5}{120} = \frac{32}{120} = \frac{4}{15} \quad \text{that is} \frac{1}{x} = \frac{4}{15}.$$

Surely if these things are equal they are also equal the other side up; accordingly we have $x = \frac{15}{4} = 3\frac{3}{4}$. Thus the combined labor of these men would finish the work in $3\frac{3}{4}$ days.

Another example: A is twice as old as B. Twenty years ago he was three times as old. Find their present ages.

If we knew the age of B twenty years ago we would know all that was necessary. So let us represent the *number* of years of B's age twenty years ago by x. Then $3x$ represents the age of A at the same time. In twenty years each of these men will become twenty years older. A's present age would therefore be represented in years by $3x+20$ and B's present age by $x+20$.

The problem says A is now twice as old as B, which fact we express in the following equation:

$$3x+20=2(x+20)$$

or $3x+20=2x+40$ Subtracting $2x+20$ from each side

we have $x=20$

$x+20=40$ (B's age now)

$3x+20=80$ (A's age now.)

It is easy to prove these solutions by inserting the obtained value for x in the problems.

5. A Different Type of Problem.—As a slightly different type of problem we will ask ourselves at what time after 2 o'clock will the hour-hand and the minute-hand of a clock be together?

The minute-hand being the more prominent of the two, we might let x equal the *number* of minute spaces that the nimute-hand travels over before the two hands come together. Now we know that between 2 and 3 o'clock the minute-hand will pass entirely round the clock, traveling 60 minute-spaces, while the hour-hand will go only from figure 2 to figure 3, covering five minute-spaces or $\frac{1}{12}$ as much as the minute-hand. Therefore if x represents the number of spaces the minute-hand travels before they get together, $\frac{x}{12}$ will represent the number of spaces the hour-hand will travel in the same time. The minute-hand will therefore gain $x-\frac{x}{12}=\frac{11}{12}x$.

Since at precisely 2 o'clock they were ten minute-spaces apart in the direction they were going, then it follows that $\frac{11x}{12}=10$. Multiplying each side by 12 we have $11x=120$, therefore

$$x=10\frac{10}{11} \text{ minute-spaces}$$

As this is the number of minute-spaces the minute-hand travels after 2 o'clock, it also represents the number of minutes after 2 when the hands are together.

Again, at what time after 5 o'clock are the hands of a clock opposite each other?

At 5 o'clock they are 25 minute-spaces apart in the direction in which they are going. To be opposite thay will have to be 30 minute-spaces apart. The minute-hand must gain 25 minute-spaces to get together and 30 minute-spaces more to be opposite.

It must gain 55 minute-spaces, so $\frac{11x}{12} = 55$

Dividing both sides by 11 we have $\frac{x}{12} = 5$

Whence, $x = 60$

At 60 minutes after 5 o'clock, that is at 6 o'clock, the hands of the clock will be opposite, which is a matter of common knowledge.

6. The Coefficient.—The number (3) in such an algebraic expression as $3x$ is called the *coefficient*. It indicates that x is to be multiplied by that number. Sometimes the coefficient is not confined to number but includes one, two, or more letters also; for instance, in $3abx$ the coefficient might be $3ab$, in which case x is to be multiplied by $3ab$. It is important to note the difference between the coefficient and the exponent, as in $3x$ and x^3. Supposing x to represent 5, then $3x = 3 \times 5 = 15$; but $x^3 = 5 \times 5 \times 5 = 125$.

7. Resemblances Between Arithmetic and Algebra.—It is self-evident that 397 can be considered as $300+90+7$. These figures can, however, also be expressed in another way, $3(10)^2+9(10)+7$. In algebra we could represent them as $3x^2+9x+7$, where x would not necessarily stand for 10 but might represent any number.

The fundamental operations of algebra are much the same as those of arithmetic, tho there are a few necessary differences. One of them is that we do

not "carry" since we do not know the value of x. In arithmetic, of course, the radix is always 10.

As in arithmetic we place *like* powers—all the x^2's, for instance—in one column. The rows are arranged in descending powers of a letter and descending alphabetical order, or in the reverse order. Such an expression as $6x^2y^3-4xy^3-4x^3y+y^4+x^4$ is never used in that form. As y is the *dominant* letter used and y^4 is the highest power of that letter, y^4 may come first and then $-4xy^3$ followed by $+6x^2y^2-4x^3y+x^4$. Thus we would write it: $y^4-4xy^3+6x^2y^2-4x^3y+x^4$. So also in the case of $x^6+x^3-x^2+2x-x^4-1$, we should write: $x^6-x^4+x^3-x^2+2x-1$. Or, it is also permissible to write it: $-1+2x-x^2+x^3-x^4+x^6$. The main thing is to be orderly and observe place. This necessity will appear still more fully in the examples of addition and subtraction given in the next section.

8. Addition and Subtraction.—In the following simple exercises in algebraic addition and subtraction it will be observed that both + and − signs are used. We learned how to interpret and operate these in the sections of Chapter VII that describe our number set of positive and negative integers.

Here are three examples in addition which fully explain themselves:

$$\begin{array}{r} 3x^2+9x+7 \\ 8x^2+4x+5 \\ \hline 11x^2+13x+12 \end{array} \qquad \begin{array}{r} 5x^2-6x+2 \\ -3x^2+4x-7 \\ \hline 2x^2-2x-5 \end{array} \qquad \begin{array}{r} \frac{1}{4}x^3-\frac{2}{3}x^2 \qquad -\frac{1}{7} \\ \frac{1}{2}x^3 \qquad +\frac{3}{5}x+\frac{1}{14} \\ \hline \frac{3}{4}x^3-\frac{2}{3}x^2+\frac{3}{5}x-\frac{1}{14} \end{array}$$

The following are examples in subtraction, equally clear and simple:

$$\begin{array}{r} 11x^2+13x+12 \\ 8x^2+\ 4x+\ 5 \\ \hline 3x^2+\ 9x+\ 7 \end{array} \qquad \begin{array}{r} 2x^2-2x-5 \\ -3x^2+4x-7 \\ \hline 5x^2-6x+2 \end{array} \qquad \begin{array}{r} \frac{3}{4}x^3-\frac{2}{3}x^2+\frac{3}{5}x-\frac{1}{14} \\ \frac{1}{2}x^3 \qquad\qquad +\frac{3}{5}x+\frac{1}{14} \\ \hline \frac{1}{4}x^3-\frac{2}{3}x^2 \qquad\quad -\frac{1}{7} \end{array}$$

There is a rule for subtraction which says: change the sign of the subtrahend and add. Thus in the second problem in subtraction this would give, changing signs, $3x^2-4x+7$ as the second row. It is better, however, to follow the regular method. In subtracting $-3x^2$ from $2x^2$ it is evident that it will take $3x^2$ to cancel the $-3x^2$ and even then we shall require $2x^2$ more to get our answer, which is $5x^2$.

9. Multiplication.—In algebraic multiplication we multiply the coefficients, and *add the exponents.* Thus x^3 times $x^2=x^5$. Likewise x^2 times $x=x^3$, and $3x^2\times 2x=6x^3$. The reason for this can be readily seen, since $x^2=x$ times x, which is also written $x.x$, using a dot to indicate times. Then $x^3=x.x.x$; hence $x^3.x^2=x.x.x.x.x=x^5$.

Bearing this in mind, and also that there is no carrying from one place to another, it will be seen from the following examples that algebraic multiplication presents no real difficulties. Thus, to multiply $3x^2+9x+7$ with $8x^2+4x+5$ we proceed in this way:

$$\begin{array}{r} 3x^2+9x+7 \\ 8x^2+4x+5 \\ \hline 24x^4+72x^3+\ 56x^2 \\ \qquad 12x^3+\ 36x^2+28x \\ \qquad\qquad\ 15x^2+45x+35 \\ \hline 24x^4+84x^3+107x^2+73x+35 \end{array}$$

Similarly:

$$\begin{array}{r} 5x^2-6x+2 \\ -\ 3x^2+4x+7 \\ \hline -15x^4+18x^3-\ 6x^2 \\ \qquad\quad 20x^3-24x^2+\ 8x \\ \qquad\qquad\quad +35x^2-42x+14 \\ \hline -15x^4+38x^3+\ 5x^2-34x+14 \end{array}$$

10. Division.—Division being the inverse of multiplication, we subtract exponents instead of adding them.

Thus, since $4x^4 \times 2x^2 = 8x^6$, therefore $8x^6 \div 2x^2 = 4x^4$. By following, step by step, the example given herewith you will see that the operation is fundamentally the same as in arithmetic together with the application of certain algebraic rules already described. It will be observed that here also there is no carrying, a circumstance which makes algebraic operations, once they are thoroly understood, actually easier than those of arithmetic. Let us divide $-15x^4+38x^3+5x^2-34x+14$ with $5x^2-6x+2$. We may arrange this as follows:

$$
\begin{array}{l}
-15x^4+38x^3+\ 5x^2-34x+14 \\
-15x^4+18x^3-\ 6x^2 \\
\hline
\qquad 20x^3+11x^2-34x \\
\qquad 20x^3-24x^2+\ 8x \\
\hline
\qquad\quad +35x^2-42x+14 \\
\qquad\quad \underline{35x^2-42x+14} \\
\end{array}
\qquad
\begin{array}{l}
(\ 5x^2-6x+2 \\
\hline
-3x^2+4x+7 \quad \text{(Ans.)}
\end{array}
$$

Note: The quotient could also be placed above the dividend, as is customary in arithmetic.

Working this by "detached coefficients", as it is called, we would have, almost as in arithmetic,

$$
\begin{array}{rl}
 & -3+\ 4+\ 7 \quad \text{(Ans.)} \\
\hline
-15+38+\ 5-34+14 & (\ 5-6+2 \\
-15+18-\ 6 & \\
\hline
\quad 20+11-34 & \\
\quad 20-24+\ 8 & \\
\hline
\qquad 35-42+14 & \\
\qquad \underline{35-42+14} & \\
\end{array}
$$

11. Place and Order in Division.—We have already shown that *like* powers should stand in the same column, but this in some problems of division creates a little difficulty in the matter of place. The rules and explanations given in the first section of this chapter, however, are sufficient guide in all such cases and the operation of the following examples can be followed without difficulty.

Divide a^5-1 by $a-1$; a^3+x^3 by $a+x$, and $x^3-3xyz+y^3+z^3$ by $x+y+z$.

$$
\begin{array}{l|l}
a^5-1 & a-1 \\
\underline{a^5-a^4} & a^4+a^3+a^2+a+1 \\
a^4-1 & \\
\underline{a^4-a^3} & \\
a^3-1 & \\
\underline{a^3-a^2} & \\
a^2-1 & \\
\underline{a^2-a} & \\
a-1 & \\
\underline{a-1} &
\end{array}
\qquad\qquad
\begin{array}{l|l}
a^3+x^3 & a+x \\
\underline{a^3+a^2x} & a^2-ax+x^2 \\
-a^2x+x^3 & \\
\underline{-a^2x-ax^2} & \\
ax^2+x^3 & \\
\underline{ax^2+x^3} &
\end{array}
$$

$$
\begin{array}{l|l}
x^3-3xyz+y^3+z^3 & x+y+z \\
\underline{x^3+x^2y+x^2z} & x^2-xy-xz+y^2-yz+z^2 \\
-x^2y-x^2z-3xyz & \\
\underline{-x^2y-xy^2-\ xyz} & \\
-x^2z+xy^2-2xyz & \\
\underline{-x^2z\ \ -\ xyz-xz^2} & \\
xy^2-\ xyz+xz^2+y^3+z^3 & \\
\underline{xy^2\ \ \ \ \ \ \ \ \ \ +y^3+y^2z} & \\
-xyz+xz^2-y^2z+z^3 & \\
\underline{-xyz\ \ \ \ \ \ -y^2z-yz^2} & \\
+xz^2+yz^2+z^3 & \\
\underline{\ \ xz^2+yz^2+z^3} &
\end{array}
$$

In the last of these three problems we not only had to keep the exponents of each letter in order, but also had to consider the alphabetical order of the letters. For instance, the first operation in that problem left us to subtract $x^3+x^2y+x^2z$ from $x^3-3xyz+y^3$. Inasmuch as $3xyz$ and x^2y are unlike, we had to bring both down into the remainder. Then also, y^3 and x^2z are unlike; consequently each had to be brought down, but as y^3 could not be used immediately we left its bringing down till it was required, which was after the third operation. The reason why, in the first re-

mainder, we did not place $3xyz$ immediately after x^2y is that the power of x^2 gives it priority over $3xyz$.

REVIEW.

1. Why are letters used in algebra? What can they represent?
2. Name the fundamental idea in algebra and illustrate its use.
3. What is the "coefficient"? How does it differ from the "exponent"?
4. How are algebraic expressions arranged in regard to their power?
5. Give rules for algebraic subtraction and multiplication.

CHAPTER IX

PRODUCTS AND ROOTS

1. Exponents.—In multiplication, as we learned in the previous chapter, exponents are added ($a^3 \times a^2 = a^{3+2} = a^5$), and in the inverse operation, division, the process is to subtract ($a^5 \div a^2 = a^{5-2} = a^3$). When we have such an expression as $(a^3)^2$, it is evident that what is meant is $a^3 \times a^3$, which equals $a^{3+3} = a^6$, and therefore we have the rule, already established in Section 9 of Chapter VIII, that in raising to a power we multiply the exponents. In the inverse process, represented by $\sqrt{a^6} = a^{\frac{6}{2}} = a^3$, we *find the root by dividing the exponent by the index of the root.* By agreement, the index (2) of square root is never written, but always understood.

These are the fundamental laws of exponents, to which, however, may be added one other and its inverse. Thus $(a^2b^3)^4 = (a^2)^4 (b^3)^4 = a^3b^{12}$ and conversely $\sqrt[4]{a^8b^{12}} = \sqrt[4]{a^8}\sqrt[4]{b^{12}} = a^2b^3$. That is, *the power of the product equals the product of the powers, and the root of a product equals the product of the roots.* In the process of using these principles we sometimes come upon such expressions as a^0, a^{-2}, $a^{\frac{1}{2}}$, $a^{-\frac{2}{3}}$, and the like. As these expressions are obtained thru the operation of established principles, they must be explained by these same laws.

Thus $a^0 \times a^6 = a^{0+6} = a^6$. Therefore, $a^0 = a^6 \div a^6 = 1$.

It appears from this that *any number with the exponent* 0 *is* 1.

Also $a^{-2} \times a^2 = a^0 = 1$. Now if the product of a^2

and a^{-2} is 1, evidently $a^{-2} = \frac{1}{a^{+2}}$. Thus the expression $\frac{x^{-3}y^2}{b^{-5}c^4} = \frac{b^5y^2}{x^3c^4}$. The rule is that *any factor with a negative exponent can be written in the other term of the fraction with a positive exponent.*

Likewise $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$. Hence, $a^{\frac{1}{2}}$ is one of the two equal factors of a. Therefore, *$a^{\frac{1}{2}}$ is the square root of a,* and $a^{\frac{1}{2}}$ and $\sqrt{a}$ are two ways of writing the same thing. The first way is preferable and more suggestive. Also $a^{\frac{3}{5}} \times a^{\frac{3}{5}} \times a^{\frac{3}{5}} \times a^{\frac{3}{5}} \times a^{\frac{3}{5}} = a^{\frac{15}{5}} = a^3$. Thus $a^{\frac{3}{5}}$ is one of the 5 equal factors of a^3 and can be written $\sqrt[5]{a^3}$.

In the like manner $a^{-\frac{2}{3}}$ turns out to be $\frac{1}{\sqrt[3]{a^2}}$.

The regular rules of operation apply to these as to other exponents. When we multiply, we add exponents; when we divide, we subtract them. When we raise to a power, we multiply; when we find the root, we divide. With exponents, multiplication calls for addition; division turns into subtraction; raising to a power is a matter of multiplication; finding of root is division. Every operation is simplified. We shall make a very interesting use of this simplifying power of exponents when we study logarithms, which are a special class of exponents.

2. Special Products.—The following special products should be first verified by actual multiplication, then translated into ordinary language, then *memorized*, as they are constantly needed in algebraic computations.

1. $(a+b)^2 = a^2+2ab+b^2$.

1′. $(a-b)^2 = a^2-2ab+b^2$.

2. $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

2′. $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$.

3. $(a+b)(a-b)=a^2-b^2$.

4. $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac$.

4′. $(a+b-c)^2=a^2+b^2+c^2+2ab-2bc-2ac$.

5. $(a+2)(a-3)=a^2-a-6$.

5′. $(3x-5y)(2x+3y)=6x^2-xy-15y^2$.

The first reads as follows: The square of the sum of two quantities is the sum of their squares plus twice their product.

The second reads that the cube of the sum of two quantities is the cube of the first plus three times the square of the first by the second plus three times the first by the square of the second plus the cube of the second.

The third special product can be stated in words, as the sum of two quantities multiplied by their difference equals the difference of their squares. Thus $(2x-3y)(2x+3y)=4x^2-9y^2$. This is useful sometimes in multiplying numbers; for example, 102×98 could be written $(100+2)(100-2)=10{,}000-4=9996$.

The fourth special product is very much like the old game of Virginia Reel. First, every number squares, then the first couple leads off and goes down the line. For example, $(2x-3y+5z)^2=?$. Well, first, square every number and we get $4x^2+9y^2+25z^2$; then $2x$ couples with $-3y$ giving $-12xy$ and with $5z$ making $20xz$; after which $-3y$ starts down the line, as it were, and with $5z$ we get $-30yz$; thus $(2x-y+5z)^2=4x^2+9y^2+25z^2-12xy+20xz-30yx$. It is no more difficult when there are more terms, only it takes longer to play the game.

The fifth special product comes from an examination of the ordinary multiplication of such numbers. The

first and last terms are just products of first and last terms; the middle term is called a cross product, as it comes by multiplying a by $+2$ and adding it to the product of a by -3, this giving $2a$ and $-3a$ which, added, is $-a$. The cross product in 5′ gives $9xy-8xy=xy$, hence the whole product is $6x^2-xy-15y^2$. The work is as follows:

$$\begin{array}{r} a+2 \\ a-3 \\ \hline a^2+2a \\ \quad -3a-6 \\ \hline a^2-\ \ a-6 \end{array} \qquad \begin{array}{r} 3x-5y \\ 2x+3y \\ \hline 6x^2-xy-15y^2 \end{array}$$

3. The Law of Binomial Expansion.—It will be observed that in the foregoing special products in both the square and the cube the exponents of the a's decrease while those of the b's increase. Also that the coefficient of the *second* term is the same as the exponent of the *power*, to which the binomial $(a+b)$ is to be raised. *The coefficient of the third term* is obtained by multiplying the coefficient of the second term by the exponent *of* (a) *in that term and dividing by the number of that term* (which is 2). Thus in $(a+b)^3$, the coefficient of the second term is 3, the same as the exponent of the power. *The next coefficient is obtained by multiplying* 3, *the coefficient, by* 2 (2 is the exponent of a in the second term of the expansion) *and dividing this result by* 2, *getting* 3. This rule will work successively, as is shown in the following examples:

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$. Verify by multiplying $(a+b)(a+b)(a+b)(a+b)$.

$(2x+3y)^4=(2x)^4+4(2x)^3(3y)+6(2x)^2(3y)^2+4(2x)(3y)^3+(3y)^4$.

Where coefficient $6=\frac{4\times 3}{2}$, coefficient $4=\frac{6\times 2}{3}$, etc., we have finally $16x^4+96x^3y+216x^2y^2+216xy^3+81y^4$.

We will simplify one term $4(2x)^3(3y)$ as a review of previous principles. This equals $4(2^3x^3)(3y)=4(2^3)(3)(x^3y)=4(8)(3)x^3y=96x^3y$. The completion of this expansion should never be attempted in one step. First write the form, then simplify. Had we had $(2x-3y)^4$ the result would have been $16x^4-96x^3y+216x^2y^2-216y^3+81y^4$ where every alternate sign is minus.

This method is applicable with all types of exponents, as is proved by the Binomial Theorem, generally attributed to Newton. The law has two parts, as follows: (1) The *exponents of the first term decrease while those of the second term decrease in the expansion.* (2) Coefficients are obtained in succession by *multiplying a known coefficient by the power of the first term which appears with it and dividing by the number of the term.* Thus the expansion $(a+b)^n$, when n is a positive integer

$$=1a^n+\frac{1.n}{1}a^{n-1}b+\frac{n(n-1)}{1.2}a^{n-2}b^2+\frac{n(n-1)(n-3)}{1.2.3}a^{n-3}b^3+..$$

4. Factors.—It is obvious that in the expression $ax+ay+ab$, the value of a is common to each of the three terms, or, in other words, each term is divisible by a common factor (a). Such being the case we can divide each term by a and place the quotient within brackets, leaving the divisor outside as coefficient, that is, $ax+ay+ab=a(x+y+b)$. Likewise $6x^3y^2+8x^4y^3-10x^5y^3=2x^3y^2(3+4xy-5x^2y)$.

In each of these two cases the operation performed is known as *removing a monomial factor*, or factor of one term. It is the most important of all the processes of factorization and, when possible, should be performed before any other process.

Another important process is that of *grouping*

similar terms. For example, when we have such an expression as $ax+ay+bx+by$, we first group the terms that seem to belong to each other and get $a(x+y)+b(x+y)$. It will be seen that $(x+y)$ is a factor of each and that what remains is $a+b$. Therefore

$$ax+ay+bx+by=(x+y)(a+b).$$

In an expression such as x^3+x^2+x+1, we see first of all that the first two terms are each divisible by x^2, therefore, applying the rule of bracketing the quotients and placing the divisor in front as coefficient, we get $x^2(x+1)$. The common divisor of x and 1 being 1, we also have $1(x+1)$.

Therefore

$$x^3+x^2+x+1=x^2(x+1)+1(x+1)=(x+1)(x^2+1).$$

$$\text{Similarly } 6x^3+3x^2-4x-2=3x^2(2x+1)-2(2x+1)$$
$$=(2x+1)(3x^2-2).$$

A third type of factoring is that of the perfect square. In the expression a^2-6a+9, we recognize that the first and last terms are squares of a and either $+3$ or -3. The middle term $-6a$ is twice the product of a and -3, therefore

$$a^2-6a+9=(a-3)^2.$$

In the quantity $4x^2-12xy+9y^2$, the first and last terms are the squares of $2x$ and $3y$, and the middle term is minus twice their product. Hence

$$4x^2-12xy+9y^2=(2x-3y)^2.$$

5. Factorization of Difference of Squares and of Cubes.— A process of much importance is the factorization of

difference of squares. Thus $a^2-b^2=(a-b)(a+b)$; also $9x^2-25k^2=(3x-5k)(3x+5k)$. These results can easily be proved by multiplication, and the two inverse processes give us the rule that *multiplication of the sum and the difference of any two quantities is equal to the difference of their squares*, and conversely, that *the difference of the squares of any two quantities is equal to the product of the sum and difference of the two quantities.*

It follows then that $a^2+2ab+b^2-x^2$ can be written $(a^2+2ab+b^2)-x^2$ or $(a+b)^2-x^2$, which factors into $a+b+x$ and $a+b-x$. Also $(3x+2y)^2-(2x+y)^2=(5x+3y)(x+y)$. Since the sum of $(3x+2y)$ and $(2x+y)$ is $(5x+3y)$ and their difference $(x+y)$. We never indicate operations we can easily perform.

The factorization of difference of cubes is made clearer when we consider that division and multiplication prove a^3+b^3 to be equal to $(a+b)(a^2-ab+b^2)$ and a^3-b^3 to equal $(a-b)(a^2+ab+b^2)$. The rule is that *the sum or difference of two cubes equals the sum or difference of the numbers multiplied by the sum of their squares and their product with the sign changed.*

Thus $8x^3-y^6=(2x-y^2)(4x^2+y^4+2xy^2)$, since $8x^3=(2x)^3$ and $y^6=(y^2)^3$.

A little examination will show that when x^6+y^6 is written $(x^3)^2+(y^3)^2$, that is, in difference of squares, it cannot be factored; but if we express it in difference of cubes, thus $(x^2)^3+(y^2)^3$, the factors are $(x^2+y^2)(x^4-x^2y^2+y^4)$. When, however, both methods—difference of squares and difference of cubes—are possible, preference should be given to difference of squares.

The reverse of the fifth type of product is of great value, especially in the solution of equations. Suppose we wish to factor x^2-x-2. We notice that

$x^2=x$ times x and $2=2\times1$ and put these expressions down under each other as follows:

x	2	$2x$
x	1	$1x$
		$-\ x$

Then we write the cross products x times 2, and x times 1. These must add up to $-x$, which we place below, the factors then becoming $(x-2)$ and $(x+1)$.

As a more difficult example, let us factor $6x^2-5x-6$. The factors of the first term are probably $3x$ and $2x$ and of the last term 3 and 2. We will write these factors as before.

$3x$	3	
$2x$	2	

This arrangement will not do, as the first factor is divisible by 3 and the second by 2 and $6x^2-5x-6$ is divisible by neither. Accordingly, another arrangement is necessary, so we will try the following:

$3x$	2	$+4x$
$2x$	3	$-9x$
		$-5x$

This shows us we must have $-9x$ and $+4x$. Hence the factors are $(3x+2)$ and $(2x-3)$.

A great deal of practice on this type of problem will give an expertness not obtained by the ordinary student and will repay any effort expended.

6. The Factor Theorem.—For the sake of brevity, any expression in x alone is often called $f(x)$, $F(x)$ and $Q(x)$. These symbols are read "f of x", "large F of x", and "Q of x" respectively. If $f(x)=4x^3+x^2-5x+2$, then $f(y)=4y^3+y^2-5y+2$. Also, $f(2)=4\,.\,2^3+2^2-5\,.\,2+2=28$, and $f(0)=4\,.\,0+0-5\,.\,0+2=2$.

Further, when $f(x)$ is divided by $x-a$ the remainder can be proved to be $f(a)$.

This remainder must be a constant since being of lower degree in x than the divisor, which is of the first degree, it cannot have any x in it. The quotient will be some expression in x, but as it is a different expression from $f(x)$ we will use $Q(x)$ to represent it and call the remainder R. Then, since the dividend equals the divisor multiplied by the quotient plus the remainder: $f(x)=Q(x)(x-a)+R$. As these two expressions are equal for all values of x they are equal when x equals a. Putting a in all places where x occurs, we get

$f(a)=Q(a)(a-a)+R$, which can be written:

$f(a)=R$, since $a-a=0$ and 0 times anything is 0.

It is evident that if $f(a)=0$, then $R=0$, the division is exact and $x-a$ is a factor of $f(x)$.

In the expression x^2-3x+2, the last term 2 is divisible by 2 and 1, which may be either both plus or both minus. Therefore,

If $f(x)=x^2-3x+2$

$f(1)=1^2-3+2$, which is 0 and one factor is $x-1$.

$f(2)=2^2-6+2=0$, and the other factor is $x-2$.

We will also factor x^3-3x-2, where the divisor of the last term is $+1$ or -1, $+2$ or -2.

$f(x)=x^3-3x-2$.

$f(1)=1^3-3-2=-4$ (This is not 0, hence $x-1$ is not a factor.)

$f(-1)=(-1)^3-3(-1)-2=-1+3-2=0$, hence $x-(-1)=x+1$ is a factor.

$f(2)=2^3-3(2)-2=8-6-2=0$, hence $x-2$ is a factor.

$f(-2)=(-2)^3-3(-2)-2=-8+6-2=-4$, hence $x-(-2)=x+2$ is not a factor.

Division will now show that $x^3-3x-2=(x+1)(x+1)(x-2)=(x+1)^2(x-2)$.

7. Highest Common Factor.—In removing the monomial factor in the first case of factoring, we took out the highest common factor. The H. C. F. of $6x^3y^2$, $8x^4y^3$, $-10x^5y^3$ is $2x^3y^2$. The process of finding the factor is the same as in arithmetic, and in many cases, as in arithmetic, we have to factor the different expressions.

In finding the H. C. F. of x^2+x-6 and $2x^2+2x-12$, it is evident that the factors of the first expression are $x+3$ and $x-2$. On dividing the second expression by 2, the result is x^2+x-6, which is the same expression as the first number, as we might have noted at once. It is immediately obvious therefore, that x^2+x-6 is the H. C. F. of these two numbers.

What is the H. C. F. of the expressions $a^2+3a-28$ and $5a^2-20a$? The factors 5 and a of the second expression can be discarded since they are not factors of the first. There is then left of the second expression, $a-4$, which is also a factor of the first expression and the H. C. F.

In finding the H. C. F., it is always advisable to select the easier expression first, as, for example, in $2a^4+4a^3-2a^2-4a$ and a^3+2a^2-3. Here, the second expression seems the easier. By the factor theorem $a-1$ is a factor, since $f(1)=1+2-3=0$. The other factor of the second expression will therefore be a^2+3a+3, which can be easily verified by division. Since the last figure (3) does not go into the last figure of the first expression, we need not consider this factor. Then discarding factor $2a$ from the first expression we have left a^3+2a^2-a-2, of which $a-1$ is evidently a factor by the factor theorem. Therefore $a-1$ is the H. C. F.

8. Least Common Multiple.—The process of finding the least common multiple, or L. C. M., in algebra, presents no new principle or operation from the corresponding operation in arithmetic. As in arithmetic, the least common multiple is the expression of lowest degree which is divisible by each factor without leaving any remainder.

The L. C. M. of $6x^3y^2$, $8x^4y^3$, $-10x^5y^3$ is $240x^5y^3$.

The L. C. M. of x^2+x-6 and $2x^2+2x-12$ is evidently $2x^2+2x-12$, which, as we have already seen, is exactly twice the value of x^2+x-6.

We also know, from the section on the Highest Common Factor, that the factors of $a^2+3a-28$ are $a+7$ and $a-4$, while those of $5a^2-20a$ are $5a(a-4)$. Therefore the L. C. M. of $a^2+3a-28$ and $5a^2-20a=5a(a-4)(a+7)$.

In general, the L. C. M. of two numbers equals the product of one number by the quotient of the other number by the H. C. F. Also the H. C. F. multiplied by the L. C. M. of two numbers equals their product. This, however, is true only of two numbers.

9. Square Root.—One of the special products, as we learned in Section 2 of this chapter, is that $(a+b)^2=a^2+2ab+b^2$. The inverse operation is expressed as $\sqrt{a^2+2ab+b^2}=a+b$. Every square may be considered as of the form $a^2+2ab+b^2$, which can be more conveniently and suggestively written $a^2+(2a+b)b$. When we wish to extract the square root of any given quantity we begin with the first term of the quantity, as a^2 in the present case. We readily see that the square root of a^2 is a, which, then, is the first term of the required root or answer. That leaves us with $2ab+b^2$ to consider. To find the next term of the root we first divide $2ab$ by twice the first term of the root, a, which is $2a$ and $2ab\div 2a=b$. By adding b to $2a$ we get $2a+b$, which forms the divisor of $2ab+b^2$, the quotient of which, b, is the second term of the required root. This simple process is made quite clear in the accompanying work-out.

$a^2+2ab+b^2$	$a+b$
a^2	a
$2ab$	
$2ab+b^2$	$2a+b$

When there is no remainder the problem is finished,

but should there be a remainder we know that there is a larger square in the given expression than we are able to estimate at sight. We must, then, start all over again.

10. Algebraic Example of Extracting Root.—The following example is a little more advanced than the foregoing one: Find the square root of $4x^4-12x^3+29x^2-30x+25$.

$$
\begin{array}{l|l}
4x^4-12x^3+29x^2-30x+25 & \underline{2x^2-3x+5} \quad \text{(Answer)} \\
\underline{4x^4} & \\
\quad -12x^3 & \\
\quad \underline{-12x^3+\ 9x^2} & \underline{4x^2-3x} \\
\qquad\qquad 20x^2 & \\
\qquad\qquad \underline{20x^2-30x+25} & \underline{4x^2-6x+5}
\end{array}
$$

The first term $4x^4$ we regard as we did a^2 in the preceding example, and instead of a as the first term of the required root we have $2x^2$. Subtracting $4x^4$, we bring down $-12x^3$. Doubling the first root term, we have $4x^2$ as the first term of the divisor. This is contained $-3x$ times in $-12x^3$, so $-3x$ is the second term of the root and also the second term of the divisor. Here $4x^2-3x$ corresponds to $2a+b$ of the previous example. Multiplying $4x^2-3x$, the complete divisor, by $-3x$ we have $-12x^3+9x^2$, which, when subtracted, leaves a remainder whose first term is $20x^2$. We have now to find the second divisor and as before double the root already found ($2x^2-3x$) giving $4x^2-6x$. It will be seen that $4x^2$, its first term, is contained exactly five times in $20x^2$ and as this 5 is the third term of the root it must (as in the case of $-3x$) be annexed to the doubled root, giving $4x^2-6x+5$ as the complete divisor. Multiplying this by 5 we get $20x^2-30x+25$. This when subtracted leaves no remainder

and there is a whole result. The process can be continued indefinitely if there is a remainder.

11. The Square Root of Numbers.—Let us now extract the square root of 1446.2809, thus applying to arithmetic the system we have just learned from algebra. First we *point off in figures of two places each way from the decimal point*, because the square of a number less than 10 must be a number less than 100 and the square of a number less than 100 must be a number less than 10,000. That is, we generally need twice as many figures in the square as in the square root.

The given square, 1446.2809, is really 1400+46+.28+.0009, but in the process of working it is convenient to use dots instead of zeros to call attention to place.

14′46.28′09	38.03
9	
5 46	68.
5 44	
2 28 09	76.0
2 28 09	
	76.03

The explanation is exactly as before. The largest square in 14 hundred is 9 hundred, of which the square root is 3 tens. Subtracting 9 from 14 we get 5 and then we bring down the next two figures 46. Doubling 3, we find that 6 tens is contained in 54 tens nine times. But as 9 times 69 = 621, we must use 8 instead and make it the second figure of our divisor and also our second root figure. Eight times 68 is 544, which subtracted from 546 leaves 2. Bringing down 28 we at once see that any third figure to 76 would make a divisor larger than 228, so we make that third figure a zero, add a zero to the desired root and bring down two more figures, 09. Then we have no difficulty in finding that 3 is the next required figure and also the last one. If there had been a re-

mainder we could have gone on adding to the decimal places as long as we desired or until we were left without any remainder. Should we desire say 9 figures we can work out 5 by this method, the remaining 4 being obtained by contracted division, the divisor being double the root already found, thus saving much labor.

12. Cube Root and Higher Roots.—Cube root can be worked in the same manner by the formula

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+(3a^2+3ab+b^2)b.$$

Fourth root can be worked as square root of square root or by the formula:

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4=$$
$$a^4+(4a^3+6a^2b+4ab^2+b^3)b.$$

In cube root of numbers we point off periods of three numbers and in the fourth root of numbers periods of four figures are pointed off.

Higher roots can be found in similar manner, but it is seldom that any other than square root is worked out, the others generally being obtained by logarithms.

13. Radicals.—We have already learned that the square root of 2, or $\sqrt{2}$, is an irrational number, called a radical or surd. We know also that $\sqrt{-2}$ is an imaginary number. Experience has shown that it simplifies matters to remove all square factors from under the radical sign. Thus $\sqrt{18}=\sqrt{9\times 2}=3\sqrt{2}$, which is generally understood to be $\pm 3\sqrt{2}$, that is, ambiguous or uncertain as to sign. Also $\sqrt{-18}=\sqrt{9\times -2}=3\sqrt{-2}$, likewise understood to be $\pm 3\sqrt{-2}$. The minus sign cannot be removed from the radical any more than we can eliminate the 2.

Cube factors can also be removed from under cube root signs, as, for example, $\sqrt[3]{24}$, which equals $\sqrt[3]{8\times 3}$ and hence $2\sqrt[3]{3}$. As you will observe, there is no ambiguity of sign. Also $\sqrt[3]{-24}=\sqrt[3]{-8\times 3}=-2\sqrt[3]{3}$. Note that $(-2)^3=-8$.

Everything possible should be taken from under the radical sign. For instance, $\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\pm\frac{\sqrt{6}}{3}$. It should be observed that as 9 was in the denominator, so also is its square root. For the same reason $\sqrt{-\frac{2}{3}}=$

$$\sqrt{-\frac{6}{9}}=\pm\frac{\sqrt{-6}}{3}=\pm\frac{1}{3}\sqrt{-6}.$$

In this example in fractions, it is evidently possible to arrange matters so that either the numerator or the denominator could be removed. In numerical work we generally remove the denominator from under the radical sign, or *rationalize the denominator* as it is called, for the reason that in computation it is desirable to keep the denominator (which is the divisor) an exact number. In calculus and other branches of mathematics, we often choose to rationalize the numerator.

Applying the same method to literal expressions, we find that

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{ab}{b^2}}=\pm\frac{\sqrt{ab}}{b} \text{ and that } \sqrt[3]{\frac{a}{b}}=\sqrt[3]{\frac{ab^2}{b^3}}=\frac{\sqrt[3]{ab^2}}{b}$$

It will be observed from the examples so far given that ambiguity of signs (expressed $\pm$) exists only when *square* roots are removed. There is no uncertainty with cube root. This difference applies to all even and odd roots—the even roots having ambiguity of signs, the odd ones certainty.

We use radical signs and fractional exponents interchangeably.

$\sqrt[3]{(x-y)^4}=(x-y)\sqrt[3]{(x-y)}$; also $a^{\frac{3}{4}}b^{\frac{5}{4}}=\sqrt[4]{a^3b^5}=b\sqrt[4]{a^3b}$.

The last expression could be written without hesitation by considering that $\frac{5}{4} = 1\frac{1}{4}$. Thus $(3x^{\frac{5}{3}} - 2x^{\frac{3}{3}}) = (3x\sqrt[3]{x^2} - 2\sqrt[3]{x^2})$ $= (3x-2)\sqrt[3]{x^2}$. Without the radical sign, our answer becomes $(3x-2)x^{\frac{2}{3}}$.

Here is another illustrative example in which all possible square factors are first removed from both numerator and denominator and the result simplified.

$$3\sqrt{\frac{12}{5}} - 5\sqrt{\frac{4}{15}} - 2\sqrt{\frac{5}{3}} = \frac{6}{5}\sqrt{15} - \frac{2}{3}\sqrt{15} - \frac{2}{3}\sqrt{15} =$$

$$\left(\frac{6}{5} - \frac{4}{3}\right)\sqrt{15} = -\frac{2}{15}\sqrt{15}$$

Suppose we wish to know which is the greater, $\sqrt[3]{9}$ or $\sqrt{5}$?

First we find that $\sqrt[3]{9} = 9^{\frac{2}{6}} = \sqrt[6]{81}$.

Also that $\sqrt{5} = 5^{\frac{3}{6}} = \sqrt[6]{125}$. Hence $\sqrt{5} > \sqrt[3]{9}$. The sign $(>)$ means *greater than*. Its reverse $(<)$ means *less than*.

14. Multiplication and Division of Radical Expressions.— We can add radical expressions only when *both the root index and the quantity under the radical sign are alike* in all the expressions added. We can compare them, multiply or divide them *only when the index of the root is the same.* Two examples in addition and one in comparison have already been solved and we will now attempt to solve the following one in multiplication:—

$\sqrt[5]{\frac{3}{64}} \times \sqrt[5]{\frac{2}{7}} = \sqrt[5]{\frac{3}{\cancel{64}_{32}} \times \frac{2}{7}} = \frac{1}{2}\sqrt[5]{\frac{3}{7}}$. Most mathematicians would leave this as it is, but if further computation were desired we could remove the denominator by multiplying by $\frac{7^4}{7^4}$ under the radical.

It will be necessary to obtain special products, containing these numbers, very similar to those of Section 2. For instance, it is often necessary to square a

quantity $(\sqrt{3}-\sqrt{2})$, and this can be done just as easily as we wrote $(a-b)^2=a^2-2ab+b^2$, i. e., the square of the first minus twice the product of the first and second plus the square of the second, so $(\sqrt{3}-\sqrt{2})^2=3-2\sqrt{6}+2=5-2\sqrt{6}$. Likewise since $(a-b)^3=a^3-3a^2b+3ab^2-b^3$ we see that $(\sqrt{3}-\sqrt{2})^3=(\sqrt{3})^3-3(\sqrt{3})^2\sqrt{2}+3\sqrt{3}(\sqrt{2})^2-(\sqrt{2})^3=3\sqrt{3}-9\sqrt{2}+6\sqrt{3}-2\sqrt{2}=9\sqrt{3}-11\sqrt{2}$. Also as $(a+b)(a-b)=a^2-b^2$ so $(10+\sqrt{5})(10-\sqrt{5})=10^2-(\sqrt{5})^2=100-5=95$.

We will now perform these same operations with imaginary expressions. A comparison of the results will be interesting, but we should be careful not to draw hasty conclusions.

$$(\sqrt{-3}+\sqrt{2})^2=-3+2\sqrt{-6}+2=-1+2\sqrt{-6}.$$

$$(\sqrt{-3}+\sqrt{2})^3=(\sqrt{-3})^3+3(\sqrt{-3})^2\sqrt{2}+$$

$$3\sqrt{-3}(\sqrt{2})^2+(\sqrt{2})^3=-3\sqrt{-3}-$$

$$9\sqrt{2}+6\sqrt{-3}+2\sqrt{2}=3\sqrt{-3}-7\sqrt{2}.$$

Also $(10+\sqrt{-5})\,(10-\sqrt{-5})=100-(-5)=100+5=105$. Be careful never to say $\sqrt{-5}\times\sqrt{-5}=\sqrt{25}=\pm5$, for here we have two equal factors and should replace this expression by -5 without any radical.

There is also the operation of division by these numbers. If we had $\frac{4}{\sqrt{5}-2}$ we would remember that since $(a+b)(a-b)=a^2-b^2$, then $(\sqrt{5}+2)(\sqrt{5}-2)=5-4=1$, which certainly is a more desirable number to divide by than $(\sqrt{5}-2)$. We then proceed as follows $\frac{4}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}=4(\sqrt{5}+2)$. (Multiplying both numerator and denominator by the same number does not change the value, as we remember we learned in arithmetic.)

If the divisor had been an imaginary number we would have proceeded in the same manner, that is,

$$\frac{4}{\sqrt{-5}-2} = \frac{4}{\sqrt{-5}-2} \times \frac{\sqrt{-5}+2}{\sqrt{-5}+2} = \frac{4(\sqrt{-5}+2)}{-5-4} = -\frac{4}{9}(\sqrt{-5}+2)$$

Here is another example:

$$\frac{x}{\sqrt{a-x}+\sqrt{a+x}} = \frac{x}{\sqrt{a-x}+\sqrt{a+x}} \times \frac{\sqrt{a-x}-\sqrt{a+x}}{\sqrt{a-x}-\sqrt{a+x}}$$

$$= \frac{x(\sqrt{a-x}-\sqrt{a+x})}{(a-x)-(a+x)} = \frac{x(\sqrt{a-x}-\sqrt{a+x})}{-2x}$$

$= \frac{1}{2}(\sqrt{a+x}-\sqrt{a-x})$, in which the result is decidedly an improvement.

The mere rapid reading of this section will do only little good, but copying it down on paper will make the whole matter clearer.

REVIEW.

1. What are the fundamental laws of exponents?
2. State how coefficients are obtained in Binomial Expansion.
3. How is the difference of squares factored?
4. Show how to make as easy as possible the process of finding the highest common factor.
5. Extract the square root of $4x^4 - 20x^3 + 37x^2 - 30x + 9$.
6. When can we add, multiply and divide radical expressions?

CHAPTER X

QUADRATICS AND SERIES

1. Quadratics in One Unknown Quantity.—Sometimes our information concerning a number comes in the form of an *equation of the second degree in the unknown*, or a *quadratic equation*, which is an equation containing the square of the unknown quantity but no higher power. How are we to identify such a number? Suppose we have

$6x^2-x-2 = 0$. Factoring this we obtain
$(3x-2)(2x+1) = 0$. Which means either
$3x-2 = 0$.
or $2x+1 = 0$.

Hence, $3x=2$ whence $x=\frac{2}{3}$, or $2x=-1$ whence $x=-\frac{1}{2}$. Then either $\frac{2}{3}$ or $-\frac{1}{2}$ is the number described by the original equation. It is evident that the process can be reversed, for if 2 and 3 are the roots of an equation, then $x-2$ and $x-3$ are each equal to 0.

Whence $(x-2)(x-3)=0$.
or $x^2-5x+6 = 0$.

Solve the following equations, by factoring:

$x^2+9x+8=0$. Here $(x+8)(x+1)=0$ whence $x=-8$ or -1.

$x^3-2x^2+x=0$. Here $x(x-1)(x-1)=0$ whence $x=0$ or 1 twice.

$x^4-13x^2+36=0$. Here $(x^2-4)(x^2-9)=0$ or $(x-2)(x+2)(x-3)(x+3)=0$ whence $x=+2$ or -2, or $+3$ or -3, generally written ± 2, ± 3.

$x^3+6x^2+11x+6=0$. By factor theorem -1 is a root and $x+1$ a factor by division. The other factor is $x^2+5x+6=(x+3)(x+2)$ so the roots are -1, -2, -3.

$4x(x-5)(x+7)=0$. Here the roots are 0, 5, -7.

We could prove these results by forming the equations of which these were the roots, thus if $x=-8$ a root, $x+8=0$ is a factor, also if $x=-1$, a root, $x+1=0$ is a factor. Hence $(x+8)(x+1)=0$. Multiplying out we have $x^2+9x+8=0$ etc.

2. Completing the Square.—Difficulty in factoring is often overcome by what is called completing the square. This operation is simply that of taking the square root of both sides, after the left member has been made a perfect square, and is illustrated by the following example, in which we have to solve the equation:

$$3x^2+2x=4$$

We first divide each member by the coefficient 3, so as to make the coefficient of x^2 unity. This gives

$$x^2+\frac{2}{3}x=\frac{4}{3}$$

As, according to the law of the square of the sum of two quantities, the middle term of the square is twice the product of the quantities and as the first quantity is x, then $\frac{2}{3}$ must be twice the second quantity. The last term is always the square of the second quantity. It is in this case $\left(\frac{1}{3}\right)^2=\frac{1}{9}$. Therefore, we add $\left(\frac{1}{3}\right)^2$ to each side of the equation and obtain

$$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{4}{3}+\frac{1}{9} \text{ or } \frac{13}{9}.$$

Extracting the square root, $x+\frac{1}{3}=\pm\frac{\sqrt{13}}{3}$

$$x=\frac{-1\pm\sqrt{13}}{3}$$

Here is the check in the *original* equation, as all checks should be.

$$3\left(\frac{-1\pm\sqrt{13}}{3}\right)^2+2\left(\frac{-1\pm\sqrt{13}}{3}\right)=$$

$$\frac{1\mp 2\sqrt{13}+13}{3}+\frac{-2\pm 2\sqrt{13}}{3}=\frac{12}{3}=4.$$

The $(\mp)$ used in the check is the opposite of $(\pm)$. When these signs occur together the $\begin{cases}\text{upper}\\\text{lower}\end{cases}$ sign is read with the $\begin{cases}\text{upper}\\\text{lower}\end{cases}$ sign, thus: $\pm 5\mp 3=2$ or -2, according to whether we take the upper or lower sign.

Take another case. Let us solve the equation $5x^2+3x+4=0$.

Transpose 4 to the right member. $5x^2+3x=-4.$

Divide each member by 5. $x^2-\frac{3}{5}x=-\frac{4}{5}.$

Add $(\frac{3}{10})^2$ to each member. $x^2-\frac{3}{5}x+(\frac{3}{10})^2=-\frac{71}{100}$

Extract the square root. $x-\frac{3}{10}=\frac{\pm\sqrt{-71}}{10}.$

$$x=\frac{3\pm\sqrt{-71}}{10}.$$

$$\text{Check: } 5\left(\frac{+3\pm\sqrt{-71}}{10}\right)^2-3\left(\frac{+3\pm\sqrt{-71}}{10}\right)+4=$$

$$\frac{+9\pm 6\sqrt{-71}-71}{20}-\frac{9\pm 3\sqrt{-71}}{10}+4=0.$$

Note $\sqrt{-71}\times\sqrt{-71}=-71.$

3. The General Quadratic.—The equation

$$ax^2+bx+c=0$$

is called the *general quadratic equation*, since it reduces to any given quadratic when proper values are given

to a, b and c. The solution of this general quadratic will therefore give us a formula from which we can write down the roots of any quadratic.

Dividing each of the terms by a (exactly as in the operation in completing the square) after transposing c, we obtain

$$x^2+\frac{b}{a}x=\frac{-c}{a}$$

Completing the square by adding to each side $\left(\frac{b}{2a}\right)^2$

$$\text{we have } x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}, \text{ or}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Following which, the square root is extracted

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

and we have the formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

This important formula cannot be simplified till we know the numerical values of a, b and c. Every equation must be simplified before applying the formula to its solution.

In solving the equation $\quad \frac{2}{x}+\frac{1}{x+2}=\frac{2}{5}$

Combine the terms with x, $\quad \frac{3x+4}{x^2+2x}=\frac{2}{5}$

Then clear fractions, $\quad 2x^2+4x=15x+20.$

Simplifying so as to have the first term positive,

$$2x^2-11x-20=0.$$

Here $\quad (a=+2,\ b=-11,\ c=-20.)$

Then by the formula $x=\dfrac{11\pm\sqrt{121+160}}{4}=\dfrac{11\pm\sqrt{281}}{4}.$

In practical work we would extract the square root of 281 and get the result to the required number of figures in one final result.

4. Relations Between the Roots.—If we let x_1 and x_2 represent the two roots of the quadratic from the formula, that is $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

then, as in adding, the radicals cancel, having opposite signs, the sum of these roots $x_1+x_2 = \frac{-b}{a}$ and their product $x_1x_2 = \frac{c}{a}$. (The numerators are the sum and difference of the same two numbers, hence the product is the difference of their squares.)

Now, the equation $ax^2+bx+c=0$ can be written

$$x^2+\frac{bx}{a}+\frac{c}{a}=0.$$

It should be carefully noted that the last term is now $\frac{c}{a}$, which is the product of the roots. Also that $\frac{b}{a}$, the coefficient of x (the second term) is opposite in sign from $\frac{-b}{a}$ the sum of the roots.

Let us check our answers to $2x^2-11x-20=0$.

If we make the coefficient of x^2 one, we have $x^2-\frac{11x}{2}-10=0$.

Our roots were $x_1 = \frac{11+\sqrt{281}}{4}$ and $x_2 = \frac{11-\sqrt{281}}{4}$.

Then $x_1+x_2 = \frac{11}{2}$, which is the negative of the coefficient of x.

Also $x_1 \times x_2 = \frac{121-281}{16} = \frac{-160}{16} = -10$, the last term of our equation, after it has been rearranged by making the coefficient of x^2 equal to one. This check is much easier than substituting $\frac{11+\sqrt{281}}{4}$ and $\frac{11-\sqrt{281}}{4}$ for x in our original equation.

Care should be taken not to accept any answer that would have introduced the operation of division by 0 in the original equation. *Division by 0 must be rigorously avoided in mathematics of every kind.* The only values x might have in our original equation that could bring about such an unfortunate circumstance as division by 0 are $x=0$ and $x=-2$, for the equation was $\frac{2}{x}+\frac{1}{x+2}=\frac{2}{5}$. As we did not get either of these results, there is no cause for error of this kind.

5. Nature of Roots of Quadratics.—In the quadratic $ax^2+bx+c=0$, we obtained by the formula the roots $x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$. The only difference between the two roots is in the radical part $-(\sqrt{b^2-4ac})$. If that were 0, the roots would be equal.

If b^2-4ac were positive, then we could take the square roots in real numbers and the roots would be *real.* If it were not only positive but also a perfect square we could extract the root exactly and since there would be no radicals we could say the roots were *rational,* as well as real. If b^2-4ac were a negative number the square root of it would be *imaginary* and the roots themselves would be imaginary or, as is more generally said, *complex.* In all these three cases—real, rational and complex—the roots would also be *unequal.*

In the equation $2x^2-11x-20=0$, we have $b^2-4ac=(-11)^2-4\times2(-20)=281$. Since 281 is positive, the roots are real and unequal; and since 281 is not a perfect square, the roots are also irrational.

If the equation were $x^2+6x+9=0$, we would have $b^2-4ac=(6)^2-4\times9=0$. That is, the roots would be equal, which means that x^2+6x+9 is a perfect square.

Equation $x^2+3x+9=0$ gives $b^2-4ac=(3)^2-4\times9=-27$,

hence the roots are imaginary or complex. In the equation, $6x^2-5x-6=0$, we have $b^2-4ac=25+144=169$, a perfect square. Hence the roots are rational.

It will be seen, therefore, that we can determine the character of the root of a quadratic without completely solving it. This not only helps us as a check on our work but is also of great importance in many other problems. Such problems as this will occur: For what values of k will the roots of the following equation be equal?

$$x^2+4kx+4=0.$$

Applying the quadratic formula for $ax^2+bx+c=0$, we have here $a=1$, $b=-4k$ and $c=4$, hence $b^2-4ac=16k^2-16$. In order that the roots be equal, we must have $16k^2-16=0$. Solving this equation for k, we get $k=\pm 1$. That is, when k in this equation has the value of 1, or -1, the equation has equal roots. We can readily verify this result. When $k=1$, the equation is $x^2-4x+4=0$ (a perfect square) and 2 is the only root; when $k=-1$, the equation is $x^2+4x+4=0$ (a perfect square), and -2 is the only root.

6. Higher Equations Worked Out Like Quadratics.—The following equations are of exceptional importance. They are not difficult and could be made still easier tho longer. Economy of work is one of the first principles of mathematics.

Solve: $c^4-13c^2+36=0$.

It should be noted that c^4 is the square of c^2 and that the expression is a quadratic in c^2. Solve for c^2, then for c.

Solve: $x^6-3x^3-4=0$. Put $y=x^3$ and we have
$y^2-3y-4=0$.

Whence $y=4$ or -1

Hence $x^3=4$ or -1 and

$x^3-4=0$ or $x^3+1=0$

Factoring $x^3+1=(x+1)(x^2-x+1)=0.$

Whence $x=-1$ or $\frac{1\pm\sqrt{-3}}{2}$

We could factor x^3-4 similarly; but it is easier to note that when $x^3=4$, $x=\sqrt[3]{4}=\sqrt[3]{4}\times\sqrt[3]{1}$, therefore we will obtain the three cube roots of 1 and multiply each of them by the arithmetical cube root of 4.

By factoring $x^3-1=(x-1)(x^2+x+1)=0$, we get

$x=1$ or, by the formula from the second part $\frac{-1\pm\sqrt{-3}}{2}$

(Notice that either one of these imaginary roots of 1 is the square of the other. They are called ω and ω^2. ω is the Greek letter omega.)

Hence $x=\sqrt[3]{4}$, or $\sqrt[3]{4}\frac{(-1+\sqrt{-3})}{2}$, or $\sqrt[3]{4}\frac{(-1-\sqrt{-3})}{2}$

(Other irrational roots are obtained in the same way from the roots of 1 or -1.)

Solve: $(x^2-3x+1)(x^2-3x+2)=12$. If y is put for x^2-3x+1

we have $y(y+1)=12$

whence $y^2+y-12=0.$

$y=3$ or $-4.$

When $x^2-3x+1=3$, $x=\frac{3\pm\sqrt{17}}{2}$

When $x^2-3x+1=-4$, $x=\frac{3\pm\sqrt{-4}}{2}$

As another type of example

solve: $\frac{1}{x^2+1}+\frac{1}{x^2+2}=\frac{1}{x^2+3}$. This becomes

$\frac{1}{y}+\frac{1}{y+1}=\frac{1}{y+2}$ when x^2+1 is put $=y$, etc.

Solve: $\frac{x-a}{x^2+a^2}+\frac{x^2+a^2}{x-a}=\frac{34}{15}$. This becomes $y+\frac{1}{y}=\frac{34}{15}$

where $\frac{x-a}{x^2+a^2}=y$ and $=\frac{x^2+a^2}{x-a}=\frac{1}{y}$.

7. Approximation.—Many of the problems which occur in physics and geometry give rise to quadratic equations. In general, the roots of such quadratics are irrational numbers appearing in the form of radical expressions. For practical purposes we usually require rational results which give approximately the values of the roots. The method used in obtaining such approximations may be seen from the following:

Approximate to two decimal places the roots of

$$\frac{25x+2}{x+1}+1-\frac{47x+31}{2x-3}=0.$$

Simplifying, we have $x^2-30x-8=0$.

$$x=\frac{30\pm\sqrt{932}}{2},$$

whence $x=\pm 15\sqrt{233}$;

that is $x=15\pm 15.265-=30.265-$ or $0.265-$.

Therefore, the roots of the given equations correct to two places of decimals are 30.26 and 0.26. As the sum of these two roots is approximately 30 and their product approximately -8, we can be satisfied, for the only numbers threatening division by 0 were $x=-1$ and $x=\frac{3}{2}$.

The solution of *homogeneous* equations in two variables, that is, of equations all of whose *terms are of the same degree*, is as readily performed as the solution of equations in one variable. The equation $x^2+4xy+3y^2=0$ is homogeneous.

To illustrate the method, we will solve $x^2+4xy=-3y^2$ by completing the square and by formula.

First, complete the square by adding $4y^2$ to each side, which gives $x^2+4xy+4y^2=y^2$

Whence $x+2y=\pm y$, and

$x=-y$ or $-3y$.

The solution by formula is $x = \frac{-4y \pm \sqrt{16y^2 - 12y^2}}{2}$

Hence $x = \frac{4y \pm 2y}{2}$

$x = -y$ or $-3y$.

8. Irrational Equations.—The unknown in an equation sometimes occurs in expressions which are under the radical sign. If $\sqrt{x} = a$, x will equal a^2. Had we not known or noticed the original expression ($\sqrt{x} = a$) we might have said $\sqrt{x} = \pm a$, which is an ambiguity and therefore to be avoided. We consider only the *positive square root* or *principal square root* in radical equations in elementary mathematics.

Solve: $\sqrt{x^2-9} - 4 = 0.$

$\sqrt{x^2-9} = 4.$

$x^2 - 9 = 16.$

$x^2 = 25.$

$x = \pm 5.$

It is absolutely necessary to test these results, both of which *happen* to prove in the original equation.

In solving $\sqrt{x-6} + 4 = 0$ we get

$\sqrt{x-6} = -4.$ As this does not express a *principal value* of the radical, we need not go any farther, but working the problem out by way of demonstration and squaring we have

$x - 6 = 16.$

Whence $x = 22.$ Substituting in the original equation we get $\sqrt{22-6} + 4 = \sqrt{16} + 4 = 4 + 4 \neq 0.$ This equation, then, has no roots.

Note.—The sign $\neq$ means *is not equal to.*

But take another equation for solution, say

$$\sqrt{x+5} - \sqrt{7x+4} + \sqrt{2x+9} = 0.$$

By arranging the terms so that one radical shall stand alone in one member, we get $\sqrt{x+5} + \sqrt{2x+9} = \sqrt{7x+4}$. Squaring

$x + 5 + 2\sqrt{(x+5)(2x+9)} + 2x + 9 = 7x + 4.$

Simplifying $2x - 5 = \sqrt{(2x+9)(x+5)}$

Again squaring and simplifying $2x^2-39x-20=0$.

Factoring $(2x+1)(x-20)=0$

giving roots 20 or $-\frac{1}{2}$. Substituting 20 for x in the original equation, we get $\sqrt{25}-\sqrt{144}+\sqrt{49}$, which simplified $=5-12+7=0$, which is an identity. Hence 20 is a root.

But substituting $-\frac{1}{2}$ in the given equation, we have

$$\sqrt{\frac{9}{2}} - \sqrt{\frac{1}{2}} + \sqrt{8} = 3\sqrt{2} \neq 0.$$

Hence $-\frac{1}{2}$ is not a solution.

Is it a solution of $\sqrt{7x+4}+\sqrt{x+5}-\sqrt{2x+9}=0$?

Is it a solution of $\sqrt{7x+4}+\sqrt{x+5}+\sqrt{2x+9}=0$?

Is it a solution of $\sqrt{7x+4}-\sqrt{x+5}+\sqrt{2x+9}=0$?

In working these last three equations, which should be done in full, we shall seem to get the same result in each case. But on attempting to verify, we arrive at some remarkable conclusions, a circumstance which emphasizes the uncertainty of an answer to a radical equation until it has been verified.

9. Exercises in Equations.—All the methods of solving equations are merely the scientific narrowing down of the possible numbers among which the result will lie. *The answer or root of an equation is that quantity which when substituted for the unknown satisfies the equation.* Everything else in the solution is but a means toward this end. The eminent French mathematician, Lagrange, said "Algebra is generous; it often gives us more than we ask for."

Let us suppose that a man bought muslin for $3. If he had bought 3 yards more for the same money each yard would have cost him 5 cents less. How many yards did he buy?

Let x stand for the number of yards the man bought, then 1 yard cost $\frac{300}{x}$ cents. If he bought $x+3$ yards for the same money, the cost of each yard would be $\frac{300}{x+3}$ cents, which having a larger denominator than $\frac{300}{x}$ is the smaller of the two.

Hence $$\frac{300}{x} - \frac{300}{x+3} = 5$$

$$\frac{60}{x} - \frac{60}{x+3} = 1$$

$$60\left(\frac{1}{x} - \frac{1}{x+3}\right) = 1$$

$$60\left(\frac{3}{x(x+3)}\right) = 1$$

$$180 = x^2+3x$$

$$x^2+3x-180 = 0$$

Whence, $x = 12$ or -15

Of the two answers, $x=12$ is the only one that conveys any meaning to us. Restate the problem, making it read that if the man had bought 3 yards less for the same money, each yard would have cost him 5 cents more. And note the changes in result.

10. Series.—An expression whose succession of terms is obtained by a fixed law is called a *series*. The express $(x+y)^3=x^3+3x^2y+3xy^2+y^3$ is a *finite* series because its law of formation does not give any more terms. The series 1, 3, 5, 7, etc., is an *infinite* series, because we can get terms forever by continuing to add 2. This last series is also called an *arithmetical series*, but a series such as 2, 4, 8, 16, 32, etc., is a *geometric series*, its terms being obtained by multiplying each by 2 to obtain the next. It is also infinite.

In arithmetical series, there is a common difference, d.

In geometric series, there is a common ratio, r.

Of the following series which are arithmetical, which geometric and which neither? There are two of each set:

(1) $-4, 0, 4, 8.$ (2) $1, -1, 1, -1, 1.$ (3) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}.$

(4) $.3, .03., .003.$ (5) $20, 13, 6, -1.$ (6) $1, 2, 5, 26.$

In series (1) and (5) there are two values of d, namely, 4 and -7. These are arithmetical series.

In series (2) and (4) there are two values of r, namely, -1 and .1. These are geometric series.

In series (3) we have terms that are reciprocals ($\frac{1}{d}$ = the reciprocal of d) of the terms of an arithmetical progression. Such a series is called a *harmonic series*. The remaining series (6) was obtained by squaring each term and then adding 1 to get the next term. It is an example of innumerable series which can be devised for experience and practice, regarding which it should be mentioned that it is easier to invent a series than to find out its law afterwards. The law should not be accepted as true until it has been tested in at least three terms.

11. Arithmetical Progression.—If the terms of an arithmetical series are represented by a, $a+d$, $a+2d$, $a+3d$, $a+4d$. . . . the 10th term will evidently be $a+9d$. In like manner the nth term will be $a+(n-1)d$. If we call the nth term l, the formula will be $l=a+(n-1)d$. Thus the 8th term of the series 7, 10, 13, 16, can be obtained by saying $l=7+(8-1)3$, whence $l=28$, which may be tested by continuing the series.

What is the common difference when 27 is the first of 12 terms of an arithmetical progression and -17 the last?

Evidently, $-17=27+(12-1)d$

Whence $d=-4$

The terms are 27, 23, 19, 15, 11, 7, 3, -1, -5, -9, -13, -17.

Insert four arithmetical means between 7 and 20. Here $n=6$ and $20=7+(5d)$

$$d=2\frac{3}{5}$$

The terms are $7, 9\frac{3}{5}, 12\frac{1}{5}, 14\frac{4}{5}, 17\frac{2}{5}, 20.$

If we let s represent the sum of an arithmetical progression, and as before a = the first term, l = the last term, d = the common difference and n = the number of terms, we shall obtain the following terms as the two ends of the series.

$s = a+(a+d)+(a+2d)+\ldots\ldots+(l-2d)+(l-d)+l$, or, written backwards
$s = l+(l-d)+(l-2d)+\ldots\ldots+(a+2a)+(a+d)+a$ Adding, we get
$2s = (a+l)+(a+l)+(a+l)+\ldots+(a+l)+(a+l)+(a+l)$.

Hence $2s = n(a+l)$, since n equals the number of terms; therefore, $s = \frac{n}{2}(a+l)$. By substituting the value of l, in $l = a+(n-1)\,d$, we get $s = \frac{n}{2}\left[2a+(n-1)d\right]$.

Thus the number of times (s) a clock strikes in twelve hours is $s = \frac{12}{2}(1+12) = 78$.

12. Geometric Progression.—The general form of the geometric progression is a, ar, ar^2, ar^3, ar^{n-1}, the increase (or decrease) being by constant ratio. Hence we say $l = ar^{n-1}$. If any three of these four letters are given the remaining one can easily be found.

Suppose the twelfth term of a G. P. to be 1536 and the fourth term 6. What are the ratio and the series?

Here $ar^{11} = 1536$ and $ar^3 = 6$. Dividing the former by the latter we get $r^8 = 256$, and, therefore, $r = \sqrt[8]{256}$ or 2. As $r = 2$, then the fourth term must be $ar^3 = 8a$, and as $8a = 6$, therefore $a = \frac{3}{4}$ and the series is $\frac{3}{4}$, $\frac{3}{2}$, 3, 6, etc.

$$\text{The sum} = s = a+ar+ar^2+ar^3+\ldots\ldots+ar^{n-1}$$
$$rs = \quad ar+ar^2+ar^3+\ldots\ldots+ar^{n-1}+ar^n$$
subtracting, $s-rs = a-ar^n$ (all the middle terms cancel)

$s = \frac{a-ar^n}{1-r}$ or $\frac{ar^n-a}{r-1}$ the latter being used when r is greater than 1. Remembering that $ar^{n-1} = l$, we note that $ar^n = rl$ and

$$s = \frac{a - rl}{1 - r} \text{ or } \frac{rl - a}{r - 1}.$$

If 3 of the 5 numbers a, r, n, l, and s are known, these two equations for l and s will enable us to find the other two.

An important application of the same formula presents itself when r is less than 1, for

$s = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$ becomes

$s_\infty = \frac{a}{1 - r}$, which means that the *sum to infinity* equals $\frac{a}{1 - r}$.

The term $\frac{ar^n}{1 - r}$ is an infinitesimal, for since r is less than 1, r^2 is less than r, etc. For example, if r is $\frac{1}{2}$, $r^2 = \frac{1}{4}$, $r^3 = \frac{1}{8}$, thus r^n becomes smaller and smaller as n grows larger and so does the term in which it stands.

13. Circulating Decimals.—Sometimes we get a decimal in which the figure or group of figures repeats itself again and again, as .333..... generally written $.\dot{3}$ (The dot over the 3 means it is repeated indefinitely.) Another example is .135135135... generally written $.\dot{1}3\dot{5}$.

The first of these numbers $.\dot{3}$ is evidently .3 + .03 + .003, etc., where $a = .3$ and $r = .1$. Hence $s_\infty = \frac{a}{1 - r}$ gives $.\dot{3} = \frac{.3}{1 - .1} = \frac{.3}{.9} = \frac{1}{3}$. In the same way it can be shown that $.\dot{1}3\dot{5} = \frac{135}{999}$ also $.\dot{0}3\dot{5} = \frac{35}{999}$. Any *repetend*, as it is called, can be replaced by a fraction whose numerator is the set of figures in the repetend and whose denominator consists of as many 9's as there are figures in the repetend. Thus $.\dot{4}\dot{3}\dot{4} = .4\frac{34}{99} = \frac{4\frac{34}{99}}{10} = \frac{430}{990} = \frac{43}{99}$. This is evidently true, since the repetend could be written $.\dot{4}\dot{3} = \frac{43}{99}$.

14. Geometric Series in Compound Interest.—One of the most useful applications of geometric series is in compound interest. Let p equal the number of dollars invested, r the rate per cent and t the time or number of interest payments. Then

$a = p(1+r)$ at the end of one year. ($a = p$ plus interest)
$a = p(1+r)^2$ at the end of two years.
$a = p(1+r)^t$ at the end of t years.

In what time will \$8000 amount to \$12,500, the rate being $3\frac{1}{2}\%$ compounded annually?

Here $\$12{,}500 = \$8000\ (1.035)^t$

Hence $\frac{12{,}500}{8000} = (1.035)^t$

Hence, by trial $t = 13$ years. Answer.

These problems are usually worked by a table of compound interest or by the use of logarithms.

What is the present value of \$2500 due in 4 years, money being worth $3\frac{1}{2}\%$ and the interest compounded semi-annually?

There are 8 periods for which the interest is $1\frac{3}{4}\%$, hence the formula becomes $\$2500 = a(1.0175)^8$. Multiplying the number in parenthesis by itself until we get the 8th power and dividing this into 2500 we find that $a = \$2176.2$.

15. Permutations.—If a man has 3 pairs of trousers, 4 coats and 5 vests, he can wear a different combination of vest and trousers on $5 \times 3 = 15$ successive days. Any one of his 4 coats could be worn with any one of these 15 different ways of wearing trousers and vests. Hence, this rather modest outfit would allow him to dress in a different manner for each of $15 \times 4 = 60$ days. Any one of these sixty possible arrangements is called a *permutation*.

Five of the members of a basket-ball team could be

arranged in $5\times4\times3\times2\times1=120$ ways, if each man were tried in each of five positions. Four men could be stood in a straight line $4\times3\times2\times1=24$ ways, this expression being called *factorial* 4, written 4!. But these four men can form a ring in only $3\times2\times1=6$ ways, for the position of the first man is immaterial, the second man can take his place in forming a ring in only one way, the third man can go between them in just two ways, while the fourth man can do so in only three ways.

Three keys can be placed upon a ring in only one way, but three men can arrange themselves in a ring in two ways. With the keys, however, we can turn the ring upside down and get what corresponds to each of the two ways the men can be arranged.

In how many ways can signals be made with eight flags, using three at a time?

The first flag can be chosen in 8 ways, the second in 7 and the third in 6. Hence the signals can be made with $8\times7\times6=336$ different arrangements, 3 flags at a time. The abbreviation for the number of ways 8 things can be taken 3 at a time is written $8P3$.

In how many ways can the letters of the word Pennsylvania be written or arranged?

There are twelve letters and if these were all different they could be arranged in 12! different ways, but there are 3 n's and 2a's. All the ways in which the n's could be arranged are just alike and would look like one way, and it is the same with the a's. Therefore, the number of ways Pennsylvania's twelve letters can be written is $\frac{12!}{3!\times2}$. There must, of course, be no canceling until the multiplication has been indicated, thus,

$$\frac{12.11.10.9.8.7.6.5.\overset{2}{\cancel{4}}.3.2.1}{3.2.1.2.1}=39{,}916{,}800 \text{ ways.}$$

16. Combinations.—When we are interested only in the group and not in the order or permutation of the individuals in the group, we speak of the group as a

combination. For example, *ab* is one combination of two of the three letters *a*, *b* and *c*, the other two possible combinations being *ac* and *bc*. The abbreviation for the combination of say, 6 things taken 4 at a time, is $6C4$. As there are, in this case, four things in each combination and as these four things can be arranged in 4!, that is $4\times3\times2\times1=24$ ways, the whole number of arrangements is equal to the number of groups or combinations multiplied by the number of arrangements in each group. Hence we say, in the language of algebra

$$6C4\times4!=6P4$$

Therefore $$6C4=\frac{6P4}{4!}=\frac{\overset{3}{\cancel{6}}.5.\cancel{4}.\cancel{3}}{\cancel{4}.\cancel{3}.2.1}=15.$$

In general $nCr\times r!=nPr$. Therefore $nCr=\frac{nPr}{r!}$. If both numerator and denominator are multiplied by $(n-r)!$ we get $nCr=\frac{n!}{r!(n-r)!}$

Whence $nC(n-r)=nCr$. Thus $11C7=11C4$, for every time we take a different 7 we take a different 4, also

$$11C7=\frac{11\times10\times9\times8\times7\times\cancel{6}\times\cancel{5}}{7\times\cancel{6}\times\cancel{5}\times4\times3\times2\times1}=11C4$$

Thus $100C98=100C2=\frac{100\times99}{1\times2}=4950.$

How many committees each consisting of 5 republicans and 4 democrats can be chosen from 20 republicans and 16 democrats.

The 5 republicans can be chosen in $20C5$ ways, and the 4 democrats in $16C4$ ways. As any set of republicans can go with any set of democrats the result is

$$20C5\times16C4=28,217,280.$$

How many arrangements of 4 consonants and 3 vowels can be made from 8 consonants and 5 vowels? We can choose them in $8C4\times5C3$ ways. As in any combination there are 7 letters, these letters can be arranged in 7! ways. The answer then is

$$8C4\times5C3\times7!=3,528,000.$$

17. Binomial Theorem.—We are now in a position to prove the binomial theorem for raising binomials to any positive integral power. Consider the product

$$(a+b)(a+b)(a+b)$$

When this is multiplied out we obtain a term in a^3 made up of the products of the first term of the factors, a term in a^2b made up of two of the first terms and one of the second, a term in ab^2 made up of one first term and two second, and a term in b^3 made up of the last three terms. The result of the multiplication is

$$a^3+3a^2b+3ab^2+b^3 \text{ for}$$

the product of the first term can be made in only one way, but we can choose two first terms out of three first terms in $3C2=3C1$ ways, exactly the same number as the ways in which we can select one last term. Hence there must be three (a^2b)'s. The same course of reasoning shows us that there are three (ab^2)'s and one (b^3), that is,

$$(a+b)^3=a^3+3C1a^2b+3C2ab^2+b^3$$

$$\text{also } (a+b)^4=a^4+4C1a^3b+4C2a^2b^2+4C3ab^3+4C4b^4$$

$$\text{and } (a+b)^n=a^n+nC1a^{(n-1)}b+nC2a^{(n-2)}b^2+nC3a^{(n-3)}b^3$$
$$+nC4a^{(n-4)}b^4$$

$$\text{or } (a+b)^n=a^n+na^{(n-1)}b+\frac{n(n-1)}{2!}a(n-2)b^2+\ldots$$
$$+\frac{n(n-1)\ldots(n-r+1)}{r!}a^{n-r}b^r+\ldots+b^n$$

Let us write the first five terms of $(1+\frac{1}{n})^n$

They are $1+nC1(1)^{(n-1)}\left(\frac{1}{n}\right)+nC2(1)^{(n-2)}\left(\frac{1}{n}\right)^2+$
$nC3\left(\frac{1}{n}\right)^3+nC4\left(\frac{1}{n}\right)^4+\ldots$

(Note: We cease to write powers of 1; they are all equal to 1.)

This expression equals

$$1+\frac{n}{1}\frac{(1)}{(n)}+\frac{n(n-1)}{1\times 2}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}\left(\frac{1}{n}\right)^3+$$
$$\frac{n(n-1)(n-2)(n-3)}{1\times 2\times 3\times 4}\left(\frac{1}{n}\right)^4+\ldots$$

Likewise

$$(1+a)^a=1+a(a)+\frac{a(a-1)}{1\times 2}a^2+\frac{a(a-1)(a-2)}{1\times 2\times 3}a^3+\text{etc.}$$

We could obtain the square root of 3, equals $3^{\frac{1}{2}}$, equals $(2+1)^{\frac{1}{2}}$, by this expansion, but it is simpler to make 1 the first term of the expansion. Thus we get

$$(2+1)^{\frac{1}{2}}=\left[2(1+\tfrac{1}{2})\right]^{\frac{1}{2}}=2^{\frac{1}{2}}(1+\tfrac{1}{2})^{\frac{1}{2}}.$$

Let us compute the $\left(1+\frac{1}{2}\right)^{\frac{1}{2}}$ and multiply our result by $\sqrt{2}$. First we simplify all expressions, like $\frac{1}{2}-1=-\frac{1}{2}$, $\frac{1}{2}-2=-\frac{3}{2}$, etc. Then $\left(1+\frac{1}{2}\right)^{\frac{1}{2}}=$

$$1+\frac{1}{2}\left(\frac{1}{2}\right)+\frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}\left(\frac{1}{2}\right)^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\times 2\times 3}\left(\frac{1}{2}\right)^3+$$
$$\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})-\frac{5}{2}}{1\times 2\times 3\times 4}\left(\frac{1}{2}\right)^4,\ \text{etc.}=$$
$$1+\frac{1}{2^2}-\frac{1}{2^5}+\frac{1}{2^7}-\frac{5}{2^{11}}+\frac{7}{2^{13}},\ \text{etc.}$$

In reducing this expression we obtain one term from another by noticing that 2^2 into 2^5 goes 2^3 times $=8$ times and that all denominations are powers of 2. As the numerators follow no law, we will multiply by them separately. The plus terms are placed in one column and the minus terms in another. The first divisor is $2^2=4$; the second is $2^5\div 2^2=2^3=8$; the third is $2^7\div 2^5=+2^2=4$, and so on, as shown in the following computation.

	+	−
$4)\underline{1.0000} \times 1 =$	1.0000	
$8)\underline{0.2500} \times 1 =$	0.2500	
$4)\underline{0.0313} \times 1 =$		0.0313
$16)\underline{0.0078} \times 1 =$	0.0078	
$4)\underline{0.0005} \times 5 =$		0.0025
$0.0001 \times 7 =$	0.0007	
	1.2585	− 0.0338 = 1.2247

$\sqrt{2} = 1.4142$

	1.2247
	4899
	122
	49
	2
$\sqrt{3} =$	1.7319

It is more accurate to multiply by the contracted method.

As we are not sure of the last figure we write $\sqrt{3} = 1.732$, which agrees with our previous value.

18. Convergency.—Had we tried $(1+2)^{\frac{1}{2}}$, we would have obtained a series with ascending powers of 2 and the longer we worked the farther away from our answer we would have gone. We cannot expand to an infinite series by the binomial theorem unless the second term is less than the first. Another example where care is needed is the following:

The expression $\frac{1}{1-x}$ gives by long division $1+x+x^2+x^3$. When x is less than 1, say, $x = \frac{1}{2}$, this becomes $\frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$ on the left and $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ which evidently has a limit, 2, on the right. *But*, when $x = 2$, $\frac{1}{1-2} = \frac{1}{-1} = -1$ on the left and on the right we have $1+2+4+8$, etc. Certainly $1+2+4+8$, etc., does not equal -1. *It is evident that these infinite series obtained by the ordinary expansions of algebra are true only under certain conditions.*

Such series as $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ whose sum has a limit are called *convergent* series and are somewhat closely related to geometric series where the ratio is less than 1. Every series whose terms increase numerically is *divergent.* Convergent series are among those whose terms decrease. Let the terms of a series be t_1, t_2, t_3, $\ldots\ldots t_n$, t_{n+1}. We will prove that if the ratio $\frac{t_{n+1}}{t_n}$ is less than r where r is less than 1, then the series is convergent. In other words, if the ratio of any term to its preceding term is less than some number r which itself is less than 1, then the series is convergent. Let $s=t_1+t_2+t_3+\ldots\ldots$ etc.

then, factoring out t_1, $s=t_1(1+\frac{t_2}{t_1}+\frac{t_3}{t_1}+\frac{t_4}{t_1}+\text{etc.})$ or multiplying both numerator and denominator by the same number, we get

$$s=t_1(1+\frac{t_2}{t_1}+\frac{t_2\times t_3}{t_1\times t_2}+\frac{t_2\times t_3\times t_4}{t_1\times t_2\times t_3}+\text{etc.})$$

Whence $s=t_1(1+\frac{t_2}{t_1}+\frac{t_2}{t_1}\times\frac{t_3}{t_2}+\frac{t_2}{t_1}\times\frac{t_3}{t_2}\times\frac{t_4}{t_3}+\text{etc.})$

As we assumed that r was greater than any of these fractions, $\frac{t_2}{t_1}$, $\frac{t_3}{t_2}$, etc., we can say replacing the fractions by r,

$$s \text{ is less than } t_1(1+r+r^2+r^3, \text{ etc.}).$$

Now $1+r+r^2+r^3$ is a geometric series with r less than 1; hence it equals $\frac{1}{1-r}$. Therefore s is less than $\frac{t_1}{1-r}$ and less than some positive finite number, since t_1, 1 and r are finite number. Inasmuch as s is positive it must be a finite number itself and the series has a limit. Such a series will evidently be convergent if some or all of its terms are negative.

Is the series $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$ convergent?

The second term of this series is $\frac{1}{2}x^2$, the third is $\frac{1}{3}x^3$, so the nth term is $\frac{1}{n}x^n$ and the next following that is $\frac{1}{n+1}x^{n+1}$. The ratio of this term to the preceding one is

$$\frac{\frac{1}{n+1}\left(x^{n+1}\right)}{\frac{1}{n}x^n} \left(= \frac{n}{n+1}x\right)$$

Dividing numerator and denominator by n we get $\left(\frac{1}{1+\frac{1}{n}}\right)x$.

When n is infinite $\frac{1}{n}$ is an infinitesimal and the ratio approaches x. When x is *less than 1 the series is convergent;* when x is *greater than 1 we have a divergent series.*

Some other method is necessary when x equals 1. We generally compare with some known series. If the ratio of the nth term of the two series is a finite number they are the same class, (both convergent or both divergent). Otherwise we learn nothing about them. The series commonly used for comparison are

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ \text{(divergent)}$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+ \text{(convergent)}$$

$$\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}- \text{(convergent when } x>1)$$

The greatest difficulty lies in writing the nth terms. This can be done only by trial and practice, but the ratio test will generally suffice for computable series.

19. Computing Value of Series.—As another example of computation we will compute the value of the series $\left(1+\frac{1}{n}\right)^n$ when n increases indefinitely.

This series was found in Section 17 of the present Chapter to be

$$1+\frac{n\,(1)}{1\,(n)}+\frac{n\,(n-1)}{1\times 2}\left(\frac{1}{n}\right)^2+\frac{n\,(n-1)\,(n-2)}{1\times 2\times 3}\left(\frac{1}{n}\right)^3+$$

$$\frac{n\,(n-1)\,(n-2)\,(n-3)}{1\times 2\times 3\times 4}\left(\frac{1}{n}\right)^4$$

which reduces to

$$1+1+\frac{(1-\frac{1}{n})}{2!}+\frac{(1-\frac{1}{n})\,(1-\frac{2}{n})}{3!}+\frac{(1-\frac{1}{n})\,(1-\frac{2}{n})\,(1-\frac{3}{n})}{4!}$$

When n is infinite, $\frac{1}{n}$ approaches 0 and the series thus becomes

$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}$ which can be computed as follows:

1)1.00000
2)1.00000
3).50000
4).16667
5).04167
6).00833
7).00139
8).00019
.00002
2.71827

Note: Dividing one factor divides the number. We can divide by n^3 by dividing three factors by n. When the denominator of a fraction is infinite, the limit of the fraction is 0. Hence $\frac{1}{n}$ etc., can be dropped in the limit. In the computation the several terms are 1 over the successive factorials. We can get each term from the preceding one. For if we have divided by 2, when we divide this quotient by 3, we will have division by $6=3!$ etc. The whole example is solved by the same labor it would have required for the last term. The sum in this example is called e. It is the base of the natural system of logarithms used in the calculus and all higher mathematics and is a very important and peculiar transcendental number.

Computors often arrange series for computation without testing whether they are convergent or not. They carry out the division to several decimal places and if in a reasonably small number of operations they get 0 in all the places in the result of a term they add up as we have done, drop the last two figures and call the result the sum of the series correct to the number of places retained.

REVIEW.

1. How is difficulty in factoring often overcome? What is the operation involved?

2. What is the special value of "the general quadratic"?

3. When and how are "approximations" obtained?

4. What is an arithmetical series? A geometric series? A harmonic series?

5. State and illustrate the meaning of "repetend" as applied to decimals.

6. Explain the respective meanings of "permutation" and "combination."

7. When is a series called "convergent"?

CHAPTER XI

LOGARITHMS

1. The Invention of Logarithms.—The extensive numerical computations required in business, in engineering, and in science were greatly simplified by the invention of *logarithms* by John Napier, Baron Merchiston (1550–1617). By means of logarithms we replace multiplication and division by addition and subtraction, likewise powers and roots by multiplication and division, thereby greatly simplifying the processes. Napier proceeded much as follows: Consider the powers of 2, both positive and negative. These exponents are in arithmetic progression while the results are in geometric progression.

x	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	etc.	A.P.
2^x	0.015625	0.03125	0.0625	0.125	0.25	0.5	1	2	4	8	16	32	64	128	256	512	1024	etc.	G.P.

If we wish to multiply 16 by 32 we add 4 and 5, the numbers above them in the A.P. line, and get 9. Looking under 9 we find 512 the product of 16 by 32. To multiply 1024 by 0.015625 we say $10 - 6 = 4$. The number under 4 is 16, their product.

To make this table really useful we would have to insert arithmetical means between the numbers in the A.P. line and geometric means between the numbers in G.P. line. Continuing this process, we can make any number appear in the G. P. line to as high a degree of approximation as is desired and at the same time can obtain the corresponding number in the A.P. line. To prepare such a table is very laborious. We give such a table for powers of 10 instead of 2, as the radix of our system of compu-

tation is 10. It is not difficult to learn to use this table, nor others that are to a higher degree of approximation but which are omitted here for lack of space.

The definition of a logarithm of a number N to a base b, where b is a positive and greater than 1, is the exponent x of the power to which the base b must be raised to produce the number N; that is, if

$$b^x = N$$

then $$x = \log_b N \text{ or } b^{\log_b N} = N$$

These three equations are of the highest importance in the treatment of logarithms and either of them implies the other. Thus, $2^5 = 32$, implies $\log_2 32 = 5$. We note the numbers in the A.P. line are the logarithms of the corresponding numbers in the G.P. line, where the base is 2.

2. Logarithms to the Base 10.—We will arrange the powers of 10 in a table somewhat similar to the powers of 2.

$10^3 = 1000$
$10^2 = 100$
$10^1 = 10$
$10^0 = 1$
$10^{-1} = .1$
$10^{-2} = .01$
$10^{-3} = .001$

It will be noticed that the logarithm of all numbers between 100 and 1000 are between 2 and 3. All numbers in this interval have 3 figures and their logarithms are 2 and some decimal. The whole number part of the logarithm is called the *characteristic*, the decimal part the *mantissa*. Thus the characteristic of all numbers of 3 figures is 2, the logarithms of all numbers between 10 and 100 are between 1 and 2. Thus the logarithms of all two-figured numbers is 1 and some decimal, the characteristic being 1. Evidently the characteristic of the logarithm of all whole numbers is

1 less than the number of figures in the whole number. The characteristic of the logarithm of 23162 is 4.

The logarithms of all decimals are evidently *negative.* In order to facilitate computation and not change the mantissa for the same set of figures we write $-1 = 9 - 10$; $-2 = 8 - 10$; $-3 = 7 - 10$ and say that the logarithms of all decimals shall have -10 after them (or a multiple of -10). Hence we can make the following rule for the characteristics of logarithms of decimals: *take the number of 0's before the first significant figure from 9 and annex* -10. Thus the logarithm of .68432 is $9..... - 10$. There are no 0's, so nothing from 9 leaves 9. The logarithm of $.068432 = 8..... - 10$, since 1, which is the number of 0's subtracted from 9, leaves 8. We now have rules for finding the characteristics of logarithms of whole numbers and decimals. They do not depend at all on the figure but only on the place where the decimal point lies.

In our table (pages 184–185) let us now look up the logarithm of 635. The *characteristic* is 2 (one less than the number of figures) and the number will be found in column marked *N*. The *mantissa* is found after 63 in the column marked 5 at the top. Thus, the logarithm 635 is 2.8028. The logarithm 63.5 is 1.8028, that of 6.35 is 0.8028, that of .635 is $9.8028 - 10$, also that of .0635 is $8.8028 - 10$. The characteristic deals only with the place. The logarithm of 635.7 is found to be 2.8033. The correction 5 for the fourth figure 7 is found in the lightly printed right-hand side of the page under 7 and in the same line in which the first two figures of the number are found. We add this correction to 8028, the mantissa for 635, getting 8033 as the mantissa for 6357.

For the sake of practice, look up as rapidly as possible the logarithms of the following 15 numbers:

log 314.5	log 5.863	log .8912
log .08765	log .8989	log 42.89
log .004627	log .08989	log .2003
log 1876000.	log .9899	log 23.23
log 6.666	log 399.8	log .03899

To begin with, look up the logarithms of these numbers without using the fourth significant figure. Then repeat, using four figures. Fifteen minutes should be amply sufficient for this test. With a little practice the time can be reduced to 10 minutes and even to 2 or 3 minutes. Don't forget to write down the characteristic before beginning to use the table, and don't forget to write down all that it is safe to write before interpolating for the last figure. After practicing on these it would be well to set up other exercises of a similar sort. In computation, accuracy and speed should go hand in hand. When one can comfortably find the logarithms of 15 numbers in 10 minutes or less, it will be time enough to attempt the reverse process.

3. **Antilogarithms.**—Suppose we wish to find the *number* when the *logarithm* is 2.8325. We need pay no attention to the decimal point until we get our figures. Looking thru the mantissas until we come to 83 for our first two figures, we run along the rows for a number 8325 and soon find this number opposite 68 in the zero column. We write the figures of our number 6800, as in a *four-place* table we always fill up *four* places. As the characteristic was 2 and as characteristic is one less than figures in whole number, we count off three figures from the beginning and write our number 680.0.

Again, the logarithm of a number is 4.8224. We cannot find 8224 in our table, the nearest numbers being 8222 and 8228. The first three figures of our number are 664. Turning to the side table of tenths

we look for the difference 2 between 8224 and the smaller number from the table, and find the 2 under 3; hence our last figure is 3, the figures of our number are 6643, and as the characteristic is 4 the number is 66430, a number of five figures.

Once more, the logarithm of a number is $9.7953 - 10$. The nearest mantissa less than 7953 is 7952 and this is opposite 62 in the column headed 4. Hence our first three figures are 624. Put them down. The difference in the table between 52 and 53 is 1. This is under either 1 or 2, either of which is our last figure. Hence our number is .6241 or .6242, a decimal.

4. Computations.—Let us say we wish to extract the $\sqrt[3]{2}$, $\sqrt[3]{3}$, and the $\sqrt[3]{6}$. We proceed as follows: $\log \sqrt[3]{2} = \frac{1}{3}\log 2 = \frac{1}{3}(.3010) = .1003$ (since logarithms are exponents). In the table of mantissas we find that .1003 stands for 1260. Hence the cube root of 2 is 1.260. Likewise $\log 3^{\frac{1}{3}} = \frac{1}{3}\log 3 = \frac{1}{3}(.4771) = .1590$, which is the logarithm of 1.442. Log $6^{\frac{1}{3}} = \frac{.7782}{3} = .2594 = \log 1.817$.

Since $\sqrt[3]{2} \times \sqrt[3]{3} = \sqrt[3]{6}$ we should be able to multiply the $\sqrt[3]{2}$ by $\sqrt[3]{3}$ and get $\sqrt[3]{6}$ and check as in the example.

1.260
1.442
1.260
504
50
2
1.816

Suppose we wish to compute the value of $\sqrt[2]{(54.04)^3 \times (376.2)^5}$. We first make a skeleton for our work.

$3 \log 54.04 = 3(1.\quad) =$

$5 \log 376.2 = 5(2.\quad) =$ _____

$2)$

$\log (\quad \text{Ans.}) =$

As logarithms are exponents, we multiply when we wish to raise to a power and divide when we wish to find a root. We will now fill in the skeleton.

$$\begin{aligned} 3 \log\ 54.04 &= 3(1.7327) = 5.1981 \\ 5 \log 376.2\ &= 5(2.5754) = 12.8770 \\ & \qquad\qquad\qquad\quad 2)\overline{18.0751} \\ \log (1090500000.\ \text{Ans.}) &= 9.0376 \end{aligned}$$

We are sure of only the first four figures, called significant figures as in our approximate computation. The characteristic 9 shows us that our first figure is the first of ten figures before the decimal place, and we give it place by supplying 0's. Had this problem been $\sqrt{(5.404)^3(3.762)^5}$ our work would have been as follows:

$$\begin{aligned} 3 \log 5.404 &= 3(.7327) = 2.1981 \\ 5 \log 3.762 &= 5(.5754) = 2.8770 \\ & \qquad\qquad 2)\overline{5.0751} \\ \log (344.8\ \text{Ans.}) &= 2.5376 \end{aligned}$$

5. Tables of Trigonometric Functions.—The other tables (pages 186–190 of this volume) are those of trigonometric functions, both natural and logarithmic. Under "Value" is given the (natural) number indicating the ratio and beside it is the logarithm of this number. The angles are also given in radians. Angles below 45° are found on the left and increase downward, the headings for these being at the top. Angles greater than 45° are found on the right. They increase upward and their headings are at the bottom.

Sin 63° 47′ is found as follows: 63° is in the lower right-hand corner of the page. Sin 63° 40′ is .8975, sin 63° 50′ is 0.8988, the difference for 10′ is 13, for 1′ = 1.3. We want the difference for 7′, which is 9.1 and may be regarded as 9. This is added to .8975, making 0.8984 = sin 63° 47′. The log sine is obtained in the same way: log sin 63° 40 = 9.9524 − 10; log sin 63° 50 = 9.9530 − 10. We get the characteristic from the natural value of the sine which has no zeros after the point and we could have found everything required by looking up 0.8984 in the table of logarithms of

numbers; but it is easier to rely upon this table. The difference for one minute is .6 and for 7 minutes 4.2 or, approximately 4. As the table is increasing we add 4 to 9.9524 giving 9.9528 as our log sin 63° 47′. All these numbers have −10 understood after them.

The log cos 35° 35′ = 9.9102. Our difference here was 4.5, which we may call 5, and we subtracted because our table was decreasing as the angle increased. These *trigonometric ratios*, sine, cosine, etc., will be explained in Trigonometry.

6. Co-Logarithms.—When numbers occur in the denominator it is a general practice to add *co-logarithms* instead of subtracting the logarithm. We can then look up all our numbers in the table and having found them, we combine them by addition alone. The idea is $\log \frac{1}{N} = \log 1 - \log N = 0 - \log N$ and the co-logarithm can then be obtained by subtracting the logarithm from 10 − 10. Thus colog 635.7 can be found as shown in the example.

In practice we do not proceed in this way, but just notice that every number subtracts from 9 except the last which subtracts from 10.

$$\begin{array}{r} 10.0000 - 10 \\ 2.8033 \\ \hline 7.1967 - 10 \end{array}$$

The logarithms in trigonometry are given to five places for greater accuracy. Such a table can be obtained at very reasonable cost from certain publishers, but is too large to be given a place in this volume. All problems can, however, be obtained with the use of our own table, but without the last figure of each number.

REVIEW.

1. What led to the invention of logarithms?
2. How are the fundamental processes simplified by logarithms?
3. Define "characteristic" and mantissa.
4. Explain why the logarithms of all decimals are negative.
5. How are trigonometric functions found from the tables?

Tables of Logarithms

N	0	1	2	3	4	5	6	7	8	9	Proportional Parts								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	16	17
3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	4	5	7	9	10	12	14	16
6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	12	14
8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7	8	10	11	12
2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	7	8	9	11	12
4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	2	4	5	6	8	9	10	11
5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	9	11
8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	6	7	7	8
8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5	6	7	7	8
9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	3	3	4	5	6	7	7
3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
N	0	1	2	3	4	5	6	7	8	9	1	2	2	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	5	5	6	7
8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6
3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3	4	4	5	6
1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3	4	4	5	6
2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3	4	4	5	6
3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5
7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	3	4
9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Trigonometric Functions

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.0000	0° 00′	.0000′	....	.0000	....	.....	.....	1.0000	.0000	90° 00′	1.5708
.0029	10	.0029	.4637	.0029	.4637	343.77	.5363	1.0000	.0000	50	1.5679
.0058	20	.0058	.7648	.0058	.7648	171.89	.2352	1.0000	.0000	40	1.5650
.0087	30	.0087	.9408	.0087	.9409	114.59	.0591	1.0000	.0000	30	1.5621
.0116	40	.0116	.0658	.0116	.0658	85.940	.9342	.9999	.0000	20	1.5592
.0145	50	.0145	.1627	.0145	.1627	68.750	.8373	.9999	.0000	10	1.5563
.0175	1° 00′	.0175	.2419	.0175	.2419	57.290	.7581	.9998	.9999	89° 00′	1.5533
.0204	10	.0204	.3088	.0204	.3089	49.104	.6911	.9998	.9999	50	1.5504
.0233	20	.0233	.3668	.0233	.3669	42.964	.6331	.9997	.9999	40	1.5475
.0262	30	.0262	.4179	.0262	.4181	38.188	.5819	.9997	.9999	30	1.5446
.0291	40	.0291	.4637	.0291	.4638	34.368	.5362	.9996	.9998	20	1.5417
.0320	50	.0320	.5050	.0320	.5053	31.242	.4947	.9995	.9998	10	1.5388
.0349	2° 00′	.0349	.5428	.0349	.5431	28.636	.4569	.9994	.9997	88° 00′	1.5359
.0378	10	.0378	.5776	.0378	.5779	26.432	.4221	.9993	.9997	50	1.5330
.0407	20	.0407	.6097	.0407	.6101	24.542	.3899	.9992	.9996	40	1.5301
.0436	30	.0436	.6397	.0437	.6401	22.904	.3599	.9990	.9996	30	1.5272
.0465	40	.0465	.6677	.0466	.6682	21.470	.3318	.9989	.9995	20	1.5243
.0495	50	.0494	.6940	.0495	.6945	20.206	.3055	.9988	.9995	10	1.5213
.0524	3° 00′	.0523	.7188	.0524	.7194	19.081	.2806	.9986	.9994	87° 00′	1.5184
.0553	10	.0552	.7423	.0553	.7429	18.075	.2571	.9985	.9993	50	1.5155
.0582	20	.0581	.7645	.0582	.7652	17.169	.2348	.9983	.9993	40	1.5126
.0611	30	.0610	.7857	.0612	.7865	16.350	.2135	.9981	.9992	30	1.5097
.0640	40	.0640	.8059	.0641	.8067	15.605	.1933	.9980	.9991	20	1.5068
.0669	50	.0669	.8251	.0670	.8261	14.924	.1739	.9978	.9990	10	1.5039
.0698	4° 00′	.0698	.8436	.0699	.8446	14.301	.1554	.9976	.9989	86° 00′	1.5010
.0727	10	.0727	.8613	.0729	.8624	13.727	.1376	.9974	.9989	50	1.4981
.0756	20	.0756	.8783	.0758	.8795	13.197	.1205	.9971	.9988	40	1.4952
.0785	30	.0785	.8946	.0787	.8960	12.706	.1040	.9969	.9987	30	1.4923
.0814	40	.0814	.9104	.0816	.9118	12.251	.0882	.9967	.9986	20	1.4893
.0844	50	.0843	.9256	.0846	.9272	11.826	.0728	.9964	.9985	10	1.4864
.0873	5° 00′	.0872	.9403	.0875	.9420	11.430	.0580	.9962	.9983	85° 00′	1.4835
.0902	10	.0901	.9545	.0904	.9563	11.059	.0437	.9959	.9982	50	1.4806
.0931	20	.0929	.9682	.0934	.9701	10.712	.0299	.9957	.9981	40	1.4777
.0960	30	.0958	.9816	.0963	.9836	10.385	.0164	.9954	.9980	30	1.4748
.0989	40	.0987	.9945	.0992	.9966	10.078	.0034	.9951	.9979	20	1.4719
.1018	50	.1016	.0070	.1022	.0093	9.7882	.9907	.9948	.9977	10	1.4690
.1047	6° 00′	.1045	.0192	.1051	.0216	9.5144	.9784	.9945	.9976	84° 00′	1.4661
.1076	10	.1074	.0311	.1080	.0336	9.2553	.9664	.9942	.9975	50	1.4632
.1105	20	.1103	.0426	.1110	.0453	9.0098	.9547	.9939	.9973	40	1.4603
.1134	30	.1132	.0539	.1139	.0567	8.7769	.9433	.9936	.9972	30	1.4573
.1164	40	.1161	.0648	.1169	.0678	8.5555	.9322	.9932	.9971	20	1.4544
.1193	50	.1190	.0755	.1198	.0786	8.3450	.9214	.9929	.9969	10	1.4515
.1222	7° 00′	.1219	.0859	.1228	.0891	8.1443	.9109	.9925	.9968	83° 00′	1.4486
.1251	10	.1248	.0961	.1257	.0995	7.9530	.9005	.9922	.9966	50	1.4457
.1280	20	.1276	.1060	.1287	.1096	7.7704	.8904	.9918	.9964	40	1.4428
.1309	30	.1305	.1157	.1317	.1194	7.5958	.8806	.9914	.9963	30	1.4399
.1338	40	.1334	.1252	.1346	.1291	7.4287	.8709	.9911	.9961	20	1.4370
.1367	50	.1363	.1345	.1376	.1385	7.2687	.8615	.9907	.9959	10	1.4341
.1396	8° 00′	.1392	.1436	.1405	.1478	7.1154	.8522	.9903	.9958	82° 00′	1.4312
.1425	10	.1421	.1525	.1435	.1569	6.9682	.8431	.9899	.9956	50	1.4283
.1454	20	.1449	.1612	.1465	.1658	6.8269	.8342	.9894	.9954	40	1.4254
.1484	30	.1478	.1697	.1495	.1745	6.6912	.8255	.9890	.9952	30	1.4224
.1513	40	.1507	.1781	.1524	.1831	6.5606	.8169	.9886	.9950	20	1.4195
.1542	50	.1536	.1863	.1554	.1915	6.4348	.8085	.9881	.9948	10	1.4166
.1571	9° 00′	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	81° 00′	1.4137
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.1571	9° 00′	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	81° 00′	1.4137
.1600	10	.1593	.2022	.1614	.2078	6.1970	.7922	.9872	.9944	50	1.4108
.1629	20	.1622	.2100	.1644	.2158	6.0844	.7842	.9868	.9942	40	1.4079
.1658	30	.1650	.2176	.1673	.2236	5.9758	.7764	.9863	.9940	30	1.4050
.1687	40	.1679	.2251	.1703	.2313	5.8708	.7687	.9858	.9938	20	1.4021
.1716	50	.1708	.2324	.1733	.2389	5.7694	.7611	.9853	.9936	10	1.3992
.1745	10° 00′	.1736	.2397	.1763	.2463	5.6713	.7537	.9848	.9934	80° 00′	1.3963
.1774	10	.1765	.2468	.1793	.2536	5.5764	.7464	.9843	.9931	50	1.3934
.1804	20	.1794	.2538	.1823	.2609	5.4845	.7391	.9838	.9929	40	1.3904
.1833	30	.1822	.2606	.1853	.2680	5.3955	.7320	.9833	.9927	30	1.3875
.1862	40	.1851	.2674	.1883	.2750	5.3093	.7250	.9827	.9924	20	1.3846
.1891	50	.1880	.2740	.1914	.2819	5.2257	.7181	.9822	.9922	10	1.3817
.1920	11° 00′	.1908	.2806	.1944	.2887	5.1446	.7113	.9816	.9919	79° 00′	1.3788
.1949	10	.1937	.2870	.1974	.2953	5.0658	.7047	.9811	.9917	50	1.3759
.1978	20	.1965	.2934	.2004	.3020	4.9894	.6980	.9805	.9914	40	1.3730
.2007	30	.1994	.2997	.2035	.3085	4.9152	.6915	.9799	.9912	30	1.3701
.2036	40	.2022	.3058	.2065	.3149	4.8430	.6851	.9793	.9909	20	1.3672
.2065	50	.2051	.3119	.2095	.3212	4.7729	.6788	.9787	.9907	10	1.3643
.2094	12° 00′	.2079	.3179	.2126	.3275	4.7046	.6725	.9781	.9904	78° 00′	1.3614
.2123	10	.2108	.3238	.2156	.3336	4.6382	.6664	.9775	.9901	50	1.3584
.2153	20	.2136	.3296	.2186	.3397	4.5736	.6603	.9769	.9899	40	1.3555
.2182	30	.2164	.3353	.2217	.3458	4.5107	.6542	.9763	.9896	30	1.3526
.2211	40	.2193	.3410	.2247	.3517	4.4494	.6483	.9757	.9893	20	1.3497
.2240	50	.2221	.3466	.2278	.3576	4.3897	.6424	.9750	.9890	10	1.3468
.2269	13° 00′	.2250	.3521	.2309	.3634	4.3315	.6366	.9744	.9887	77° 00′	1.3439
.2298	10	.2278	.3575	.2339	.3691	4.2747	.6309	.9737	.9884	50	1.3410
.2327	20	.2306	.3629	.2370	.3748	4.2193	.6252	.9730	.9881	40	1.3381
.2356	30	.2334	.3682	.2401	.3804	4.1653	.6196	.9724	.9878	30	1.3352
.2385	40	.2363	.3734	.2432	.3859	4.1126	.6141	.9717	.9875	20	1.3323
.2414	50	.2391	.3786	.2462	.3914	4.0611	.6086	.9710	.9872	10	1.3294
.2443	14° 00′	.2419	.3837	.2493	.3968	4.0108	.6032	.9703	.9869	76° 00′	1.3265
.2473	10	.2447	.3887	.2524	.4021	3.9617	.5979	.9696	.9866	50	1.3235
.2502	20	.2476	.3937	.2555	.4074	3.9136	.5926	.9689	.9863	40	1.3206
.2531	30	.2504	.3986	.2586	.4127	3.8667	.5873	.9681	.9859	30	1.3177
.2560	40	.2532	.4035	.2617	.4178	3.8208	.5822	.9674	.9856	20	1.3148
.2589	50	.2560	.4083	.2648	.4230	3.7760	.5770	.9667	.9853	10	1.3119
.2618	15° 00′	.2588	.4130	.2679	.4281	3.7321	.5719	.9659	.9849	75° 00′	1.3090
.2647	10	.2616	.4177	.2711	.4331	3.6891	.5669	.9652	.9846	50	1.3061
.2676	20	.2644	.4223	.2742	.4381	3.6470	.5619	.9644	.9843	40	1.3032
.2705	30	.2672	.4269	.2773	.4430	3.6059	.5570	.9636	.9839	30	1.3003
.2734	40	.2700	.4314	.2805	.4479	3.5656	.5521	.9628	.9836	20	1.2974
.2763	50	.2728	.4359	.2836	.4527	3.5261	.5473	.9621	.9832	10	1.2945
.2793	16° 00′	.2756	.4403	.2867	.4575	3.4874	.5425	.9613	.9828	74° 00′	1.2915
.2822	10	.2784	.4447	.2899	.4622	3.4495	.5378	.9605	.9825	50	1.2886
.2851	20	.2812	.4491	.2931	.4669	3.4124	.5331	.9596	.9821	40	1.2857
.2880	30	.2840	.4533	.2962	.4716	3.3759	.5284	.9588	.9817	30	1.2828
.2909	40	.2868	.4576	.2994	.4762	3.3402	.5238	.9580	.9814	20	1.2799
.2938	50	.2896	.4618	.3026	.4808	3.3052	.5192	.9572	.9810	10	1.2770
.2967	17° 00′	.2924	.4659	.3057	.4853	3.2709	.5147	.9563	.9806	73° 00′	1.2741
.2996	10	.2952	.4700	.3089	.4898	3.2371	.5102	.9555	.9802	50	1.2712
.3025	20	.2979	.4741	.3121	.4943	3.2041	.5057	.9546	.9798	40	1.2683
.3054	30	.3007	.4781	.3153	.4987	3.1716	.5013	9537	.9794	30	1.2654
.3083	40	.3035	.4821	.3185	.5031	3.1397	.4969	.9528	.9790	20	1.2625
.3113	50	.3062	.4861	.3217	.5075	3.1084	.4925	.9520	.9786	10	1.2595
.3142	18° 00′	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	72° 00′	1.2566
Radians	Degrees	Cosine Value	Cosine $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Sine Value	Sine $\log_{10}$	Degrees	Radians

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.3142	18° 00′	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	72° 00′	1.2566
.3171	10	.3118	.4939	.3281	.5161	3.0475	.4839	.9502	.9778	50	1.2537
.3200	20	.3145	.4977	.3314	.5203	3.0178	.4797	.9492	.9774	40	1.2508
.3229	30	.3173	.5015	.3346	.5245	2.9887	.4755	.9483	.9770	30	1.2479
.3258	40	.3201	.5052	.3378	.5287	2.9600	.4713	.9474	.9765	20	1.2450
.3287	50	.3228	.5090	.3411	.5329	2.9319	.4671	.9465	.9761	10	1.2421
.3316	19° 00′	.3256	.5126	.3443	.5370	2.9042	.4630	.9455	.9757	71° 00′	1.2392
.3345	10	.3283	.5163	.3476	.5411	2.8770	.4589	.9446	.9752	50	1.2363
.3374	20	.3311	.5199	.3508	.5451	2.8502	.4549	.9436	.9748	40	1.2334
.3403	30	.3338	.5235	.3541	.5491	2.8239	.4509	.9426	.9743	30	1.2305
.3432	40	.3365	.5270	.3574	.5531	2.7980	.4469	.9417	.9739	20	1.2275
.3462	50	.3393	.5306	.3607	.5571	2.7725	.4429	.9407	.9734	10	1.2246
.3491	20° 00′	.3420	.5341	.3640	.5611	2.7475	.4389	.9397	.9730	70° 00′	1.2217
.3520	10	.3448	.5375	.3673	.5650	2.7228	.4350	.9387	.9725	50	1.2188
.3549	20	.3475	.5409	.3706	.5689	2.6985	.4311	.9377	.9721	40	1.2159
.3578	30	.3502	.5443	.3739	.5727	2.6746	.4273	.9367	.9716	30	1.2130
.3607	40	.3529	.5477	.3772	.5766	2.6511	.4234	.9356	.9711	20	1.2101
.3636	50	.3557	.5510	.3805	.5804	2.6279	.4196	.9346	.9706	10	1.2072
.3665	21° 00′	.3584	.5543	.3839	.5842	2.6051	.4158	.9336	.9702	69° 00′	1.2043
.3694	10	.3611	.5576	.3872	.5879	2.5826	.4121	.9325	.9697	50	1.2014
.3723	20	.3638	.5609	.3906	.5917	2.5605	.4083	.9315	.9692	40	1.1985
.3752	30	.3665	.5641	.3939	.5954	2.5386	.4046	.9304	.9687	30	1.1956
.3782	40	.3692	.5673	.3973	.5991	2.5172	.4009	.9293	.9682	20	1.1926
.3811	50	.3719	.5704	.4006	.6028	2.4960	.3972	.9283	.9677	10	1.1897
.3840	22° 00′	.3746	.5736	.4040	.6064	2.4751	.3936	.9272	.9672	68° 00′	1.1868
.3869	10	.3773	.5767	.4074	.6100	2.4545	.3900	.9261	,9667	50	1.1839
.3898	20	.3800	.5798	.4108	.6136	2.4342	.3864	.9250	.9661	40	1.1810
.3927	30	.3827	.5828	.4142	.6172	2.4142	.3828	.9239	.9656	30	1.1781
.3956	40	.3854	.5859	.4176	.6208	2.3945	.3792	.9228	.9651	20	1.1752
.3985	50	.3881	.5889	.4210	.6243	2.3750	.3757	.9216	.9646	10	1.1723
.4014	23° 00′	.3907	.5919	.4245	.6279	2.3559	.3721	.9205	.9640	67° 00′	1.1694
.4043	10	.3934	.5948	.4279	.6314	2.3369	.3686	.9194	.9635	50	1.1665
.4072	20	.3961	.5978	.4314	.6348	2.3183	.3652	.9182	.9629	40	1.1636
.4102	30	.3987	.6007	.4348	.6383	2.2998	.3617	.9171	.9624	30	1.1606
.4131	40	.4014	.6036	.4383	.6417	2.2817	.3583	.9159	.9618	20	1.1577
.4160	50	.4041	.6065	.4417	.6452	2.2637	.3548	.9147	.9613	10	1.1548
.4189	24° 00′	.4067	.6093	.4452	.6486	2.2460	.3514	.9135	.9607	66° 00′	1.1519
.4218	10	.4094	.6121	.4487	.6520	2.2286	.3480	.9124	.9602	50	1.1490
.4247	20	.4120	.6149	.4522	.6553	2.2113	.3447	.9112	.9596	40	1.1461
.4276	30	.4147	.6177	.4557	.6587	2.1943	.3413	.9100	.9590	30	1.1432
.4305	40	.4173	.6205	.4592	.6620	2.1775	.3380	.9088	.9584	20	1.1403
.4334	50	.4200	.6232	.4628	.6654	2.1609	.3346	.9075	.9579	10	1.1374
.4363	25° 00′	.4226	.6259	.4663	.6687	2.1445	.3313	.9063	.9573	65° 00′	1.1345
.4392	10	.4253	.6286	.4699	.6720	2.1283	.3280	.9051	.9567	50	1.1316
.4422	20	.4279	.6313	.4734	.6752	2.1123	.3248	.9038	.9561	40	1.1286
.4451	30	.4305	.6340	.4770	.6785	2.0965	.3215	.9026	.9555	30	1.1257
.4480	40	.4331	.6366	.4806	.6817	2.0809	.3183	.9013	.9549	20	1.1228
.4509	50	.4358	.6392	.4841	.6850	2.0655	.3150	.9001	.9543	10	1.1199
.4538	26° 00′	.4384	.6418	.4877	.6882	2.0503	.3118	.8988	.9537	64° 00′	1.1170
.4567	10	.4410	.6444	.4913	.6914	2.0353	.3086	.8975	.9530	50	1.1141
.4596	20	.4436	.6470	.4950	.6946	2.0204	.3054	.8962	.9524	40	1.1112
.4625	30	.4462	.6495	.4986	.6977	2.0057	.3023	.8949	.9518	30	1.1083
.4654	40	.4488	.6521	.5022	.7009	1.9912	.2991	.8936	.9512	20	1.1054
.4683	50	.4514	.6546	.5059	.7040	1.9768	.2960	.8923	.9505	10	1.1025
.4712	27° 00′	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	63° 00′	1.0996
		Cosine Value	Cosine $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Sine Value	Sine $\log_{10}$	Degrees	Radians

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.4712	27° 00′	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	63° 00′	1.0996
.4741	10	.4566	.6595	.5132	.7103	1.9486	.2897	.8897	.9492	50	1.0966
.4771	20	.4592	.6620	.5169	.7134	1.9347	.2866	.8884	.9486	40	1.0937
.4800	30	.4617	.6644	.5206	.7165	1.9210	.2835	.8870	.9479	30	1.0908
.4829	40	.4643	.6668	.5243	.7196	1.9074	.2804	.8857	.9473	20	1.0879
.4858	50	.4669	.6692	.5280	.7226	1.8940	.2774	.8843	.9466	10	1.0850
.4887	28° 00′	.4695	.6716	.5317	.7257	1.8807	.2743	.8829	.9459	62° 00′	1.0821
.4916	10	.4720	.6740	.5354	.7287	1.8676	.2713	.8816	.9453	50	1.0792
.4945	20	.4746	.6763	.5392	.7317	1.8546	.2683	.8802	.9446	40	1.0763
.4974	30	.4772	.6787	.5430	.7348	1.8418	.2652	.8788	.9439	30	1.0734
.5003	40	.4797	.6810	.5467	.7378	1.8291	.2622	.8774	.9432	20	1.0705
.5032	50	.4823	.6833	.5505	.7408	1.8165	.2592	.8760	.9425	10	1.0676
.5061	29° 00′	.4848	.6856	.5543	.7438	1.8040	.2562	.8746	.9418	61° 00′	1.0647
.5091	10	.4874	.6878	.5581	.7467	1.7917	.2533	.8732	.9411	50	1.0617
.5120	20	.4899	.6901	.5619	.7497	1.7796	.2503	.8718	.9404	40	1.0588
.5149	30	.4924	.6923	.5658	.7526	1.7675	.2474	.8704	.9397	30	1.0559
.5178	40	.4950	6946	.5696	.7556	1.7556	.2444	.8689	.9390	20	1.0530
.5207	50	.4975	.6968	.5735	.7585	1.7437	.2415	.8675	.9383	10	1.0501
.5236	30° 00′	.5000	.6990	.5774	.7614	1.7321	.2386	.8660	.9375	60° 00′	1.0472
.5256	10	.5025	.7012	.5812	.7644	1.7205	.2356	.8646	.9368	50	1.0443
.5294	20	.5050	.7033	.5851	.7673	1.7090	.2327	.8631	.9361	40	1.0414
.5323	30	.5075	.7055	.5890	.7701	1.6977	.2299	.8616	.9353	30	1.0385
.5352	40	.5100	.7076	.5930	.7730	1.6864	.2270	.8601	.9346	20	1.0356
.5381	50	.5125	.7097	.5969	.7759	1.6753	.2241	.8587	.9338	10	1.0327
.5411	31° 00′	.5150	.7118	.6009	.7788	1.6643	.2212	.8572	.9331	59° 00′	1.0297
.5440	10	.5175	.7139	.6048	.7816	1.6534	.2184	.8557	.9323	50	1.0268
.5469	20	.5200	.7160	.6088	.7845	1.6426	.2155	.8542	.9315	40	1.0239
.5498	30	.5225	.7181	.6128	.7873	1.6319	.2127	.8526	.9308	30	1.0210
.5527	40	.5250	.7201	.6168	.7902	1.6212	.2098	.8511	.9300	20	1.0181
.5556	50	.5275	.7222	.6208	.7930	1.6107	.2070	.8496	.9292	10	1.0152
.5585	32° 00′	.5299	.7242	.6249	.7958	1.6003	.2042	.8480	.9284	58° 00′	1.0123
.5614	10	.5324	.7262	.6289	.7986	1.5900	.2014	.8465	.9276	50	1.0094
.5643	20	.5348	.7282	.6330	.8014	1.5798	.1986	.8450	.9268	40	1.0065
.5672	30	.5373	.7302	.6371	.8042	1.5697	.1958	.8434	.9260	30	1.0036
.5701	40	.5398	.7322	.6412	.8070	1.5597	.1930	.8418	.9252	20	1.0007
.5730	50	.5422	.7342	.6453	.8097	1.5497	.1903	.8403	.9244	10	.9977
.5760	33° 00′	.5446	.7361	.6494	.8125	1.5399	.1875	.8387	.9236	57° 00′	.9948
.5789	10	.5471	.7380	.6536	.8153	1.5301	.1847	.8371	.9228	50	.9919
.5818	20	.5495	.7400	.6577	.8180	1.5204	.1820	.8355	.9219	40	.9890
.5847	30	.5519	.7419	.6619	.8208	1.5108	.1792	.8339	.9211	30	.9861
.5876	40	.5544	.7438	.6661	.8235	1.5013	.1765	.8323	.9203	20	.9832
.5905	50	.5568	.7457	.6703	.8263	1.4919	.1737	.8307	.9194	10	.9803
.5934	34° 00′	.5592	.7476	.6745	.8290	1.4826	.1710	.8290	.9186	56° 00′	.9774
.5963	10	.5616	.7494	.6787	.8317	1.4733	.1683	.8274	.9177	50	.9745
.5992	20	.5640	.7513	.6830	.8344	1.4641	.1656	.8258	.9169	40	.9716
.6021	30	.5664	.7531	.6873	.8371	1.4550	.1629	.8241	.9160	30	.9687
.6050	40	.5688	.7550	.6916	.8398	1.4460	.1602	.8225	.9151	20	.9657
.6080	50	.5712	.7568	.6959	.8425	1.4370	.1575	.8208	.9142	10	.9628
.6109	35° 00′	.5736	.7586	.7002	.8452	1.4281	.1548	.8192	.9134	55° 00′	.9599
.6138	10	.5760	.7604	.7046	.8479	1.4193	.1521	.8175	.9125	50	.9570
.6167	20	.5783	.7622	.7089	.8506	1.4106	.1494	.8158	.9116	40	.9541
.6196	30	.5807	.7640	.7133	.8533	1.4019	.1467	.8141	.9107	30	.9512
.6225	40	.5831	.7657	.7177	.8559	1.3934	.1441	.8124	.9098	20	.9483
.6254	50	.5854	.7675	.7221	.8586	1.3848	.1414	.8107	.9089	10	.9454
.6283	36° 00′	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	54° 00′	.9425
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.6283	36° 00′	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	54° 00′	.9425
.6312	10	.5901	.7710	.7310	.8639	1.3680	.1361	.8073	.9070	50	.9396
.6341	20	.5925	.7727	.7355	.8666	1.3597	.1334	.8056	.9061	40	.9367
.6370	30	.5948	.7744	.7400	.8692	1.3514	.1308	.8039	.9052	30	.9338
.6400	40	.5972	.7761	.7445	.8718	1.3432	.1282	.8021	.9042	20	.9308
.6429	50	.5995	.7778	.7490	.8745	1.3351	.1255	.8004	.9033	10	.9279
.6458	37° 00′	.6018	.7795	.7536	.8771	1.3270	.1229	.7986	.9023	53° 00′	.9250
.6487	10	.6041	.7811	.7581	.8797	1.3190	.1203	.7969	.9014	50	.9221
.6516	20	.6065	.7828	.7627	.8824	1.3111	.1176	.7951	.9004	40	.9192
.6545	30	.6088	.7844	.7673	.8850	1.3032	.1150	.7934	.8995	30	.9163
.6574	40	.6111	.7861	.7720	.8876	1.2954	.1124	.7916	.8985	20	.9134
.6603	50	.6134	.7877	.7766	.8902	1.2876	.1098	.7898	.8975	10	.9105
.6632	38° 00′	.6157	.7893	.7813	.8928	1.2799	.1072	.7880	.8965	52° 00′	.9076
.6661	10	.6180	.7910	.7860	.8954	1.2723	.1046	.7862	.8955	50	.9047
.6690	20	.6202	.7926	.7907	.8980	1.2647	.1020	.7844	.8945	40	.9018
.6720	30	.6225	.7941	.7954	.9006	1.2572	.0994	.7826	.8935	30	.8988
.6749	40	.6248	.7957	.8002	.9032	1.2497	.0968	.7808	.8925	20	.8959
.6778	50	.6271	.7973	.8050	.9058	1.2423	.0942	.7790	.8915	10	.8930
.6807	39° 00′	.6293	.7989	.8098	.9084	1.2349	.0916	.7771	.8905	51° 00′	.8901
.6836	10	.6316	.8004	.8146	.9110	1.2276	.0890	.7753	.8895	50	.8872
.6865	20	.6338	.8020	.8195	.9135	1.2203	.0865	.7735	.8884	40	.8843
.6894	30	.6361	.8035	.8243	.9161	1.2131	.0839	.7716	.8874	30	.8814
.6923	40	.6383	.8050	.8292	.9187	1.2059	.0813	.7698	.8864	20	.8785
.6952	50	.6406	.8066	.8342	.9212	1.1988	.0788	.7679	.8853	10	.8756
.6981	40° 00′	.6428	.8081	.8391	.9238	1.1918	.0762	.7660	.8843	50° 00′	.8727
.7010	10	.6450	.8096	.8441	.9264	1.1847	.0736	.7642	.8832	50	.8698
.7039	20	.6472	.8111	.8491	.9289	1.1778	.0711	.7623	.8821	40	.8668
.7069	30	.6494	.8125	.8541	.9315	1.1708	.0685	.7604	.8810	30	.8639
.7098	40	.6517	.8140	.8591	.9341	1.1640	.0659	.7585	.8800	20	.8610
.7127	50	.6539	.8155	.8642	.9366	1.1571	.0634	.7566	.8789	10	.8581
.7156	41° 00′	.6561	.8169	.8693	.9392	1.1504	.0608	.7547	.8778	49° 00′	.8552
.7185	10	.6583	.8184	.8744	.9417	1.1436	.0583	.7528	.8767	50	.8523
.7214	20	.6604	.8198	.8796	.9443	1.1369	.0557	.7509	.8756	40	.8494
.7243	30	.6626	.8213	.8847	.9468	1.1303	.0532	.7490	.8745	30	.8465
.7272	40	.6648	.8227	.8899	.9494	1.1237	.0506	.7470	.8733	20	.8436
.7301	50	.6670	.8241	.8952	.9519	1.1171	.0481	.7451	.8722	10	.8407
.7330	42° 00′	.6691	.8255	.9004	.9544	1.1106	.0456	.7431	.8711	48° 00′	.8378
.7359	10	.6713	.8269	.9057	.9570	1.1041	.0430	.7412	.8699	50	.8348
.7389	20	.6734	.8283	.9110	.9595	1.0977	.0405	.7392	.8688	40	.8319
.7418	30	.6756	.8297	.9163	.9621	1.0913	.0379	.7373	.8676	30	.8290
.7447	40	.6777	.8311	.9217	.9646	1.0850	.0354	.7353	.8665	20	.8261
.7476	50	.6799	.8324	.9271	.9671	1.0786	.0329	.7333	.8653	10	.8232
.7505	43° 00′	.6820	.8338	.9325	.9697	1.0724	.0303	.7314	.8641	47° 00′	.8203
.7534	10	.6841	.8351	.9380	.9722	1.0661	.0278	.7294	.8629	50	.8174
.7563	20	.6862	.8365	.9435	.9747	1.0599	.0253	.7274	.8618	40	.8145
.7592	30	.6884	.8378	.9490	.9772	1.0538	.0228	.7254	.8606	30	.8116
.7621	40	.6905	.8391	.9545	.9798	1.0477	.0202	.7234	.8594	20	.8087
.7650	50	.6926	.8405	.9601	.9823	1.0416	.0177	.7214	.8582	10	.8058
.7679	44° 00′	.6947	.8418	.9657	.9848	1.0355	.0152	.7193	.8569	46° 00′	.8029
.7709	10	.6967	.8431	.9713	.9874	1.0295	.0126	.7173	.8557	50	.7999
.7738	20	.6988	.8444	.9770	.9899	1.0235	.0101	.7153	.8545	40	.7970
.7767	30	.7009	.8457	.9827	.9924	1.0176	.0076	.7133	.8532	30	.7941
.7796	40	.7030	.8469	.9884	.9949	1.0117	.0051	.7112	.8520	20	.7912
.7825	50	.7050	.8482	.9942	.9975	1.0058	.0025	.7092	.8507	10	.7883
.7854	45° 00′	.7071	.8495	1.0000	.0000	1.0000	.0000	.7071	.8495	45° 00′	.7854
		Value Cosine	$\log_{10}$	Value Cotangent	$\log_{10}$	Value Tangent	$\log_{10}$	Value Sine	$\log_{10}$	Degrees	Radians

CHAPTER XII

TRIGONOMETRY

1. Directed Length.—The line AB means the line from A to B. The line BA means the line from B to A. It is just the opposite of AB. We say $BA = -AB$ or $AB = -BA$. The $-$ sign always indicates opposition.

A ——— B

A	B	C	D	E
B	A	E	D	C

In either figure $AB + BC + CD + DE = AE$.

2. Measurement of Angle.—Angles are measured *contra-clockwise*, that is, opposite to the way the hands of a clock move. Angles measured in the *opposite direction*, that is, clockwise, are called *negative angles*.

There are several ways of measuring angles. The most common is by means of degrees. Possibly because the ancient Babylonians thought the sun went around the earth once every 360 days, they divided a complete revolution into 360 degrees. This method, tho artificial, is still used in most tables. In all higher mathematics a unit angle called the *radian* is used. This angle has an arc equal in length to the radius.

As geometry tells us, the circumference of a circle is just 2π lengths of its radius, hence there are 2π radians in $360°$ ($\pi = 3.1416$ approximately). From this we obtain:

π radians $= 180°$

1 radian $= \frac{180°}{\pi}$

1 degree $= \frac{\pi}{180}$ radians.

Accordingly, we can change degrees to radians by multiplying by $\frac{\pi}{180}$ radians and we can change radians to degrees by multiplying by $\frac{180}{\pi}$ degrees.

1 radian $= \frac{180°}{\pi} = 57.296$, approximately 57.3

1 degree $= \frac{\pi}{180} = .016898$ radians, approximately .017 radians.

Let L, Fig. 112, represent the length of an arc, R the length of its radius, and θ (the Greek letter *theta*) the number of radians in the angle. We evidently get this number by dividing the length of the arc by the length of the radius, there being as many radians in the angle as there are lengths of the radius on the arc. Hence we say $\frac{L}{R} = \theta$. From this, $L = R \times \theta$, and $R = \frac{L}{\theta}$.

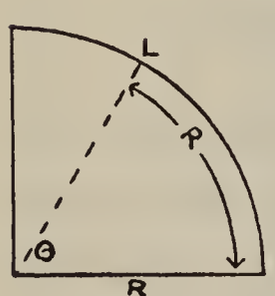

Fig. 112.

If a man walks a distance of 150 feet around a circular track whose radius is 100 feet, what angle does his final direction make with his original direction? Here $\frac{L}{R} = \frac{150}{100}$ equals the number of radians he has turned thru. This equals

$$\frac{\overset{30}{\cancel{150}}}{\underset{2}{\cancel{100}}} \times \frac{9}{\frac{\cancel{180}}{\pi}} \text{ degrees} = \frac{270}{\pi} \text{ degrees} = 85.944 \text{ degrees.}$$

Changing the decimals into minutes and seconds, we have:

0.944 the fraction of degrees $\times$ 60 minutes $= 56.64$ minutes

0.64 the fractions of minutes $\times$ 60 seconds $= 38.4$ seconds

The answer is therefore 85° 56′ 38″ .4.

3. Quadrants.—Angles may often be distinguished by quadrants in which they lie. The first quadrant extends from 0 to 90°, the second from 90° to 180°, etc. Thus, an angle of 50° is in the first quadrant, Fig. 113, an angle of 150° is in the second quadrant, an angle of 200° is in the third quadrant, an angle of 300° is in the fourth quadrant and an angle of 400° is again in the first quadrant.

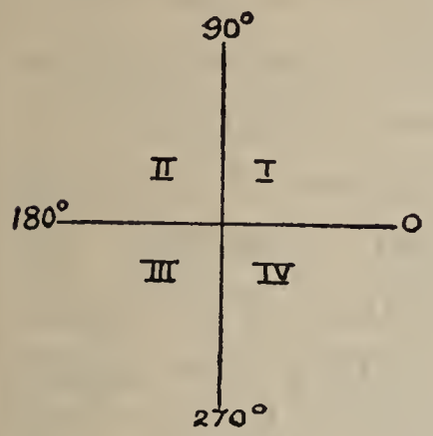

Fig. 113.

Suppose we ask in what quadrant is an angle of 1000°? We see at once 1000° is equal to two complete revolutions (720°) and 280° more. And since 280° is in the fourth quadrant, therefore 1000° is also in the fourth quadrant.

In what quadrant is − 200°? We readily see that the position of − 200° is the same as that of 160° in its terminal line. And since 160° is in the second quadrant, so also is − 200°.

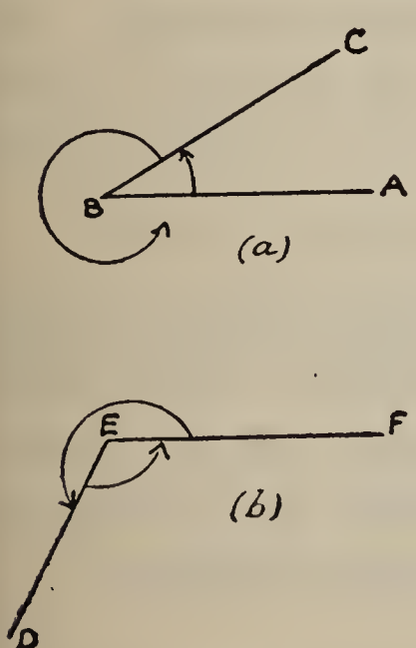

Fig. 114

The angle *ABC*, Fig. 114 (*a*), is in the first quadrant.

The angle *CBA* is in the fourth quadrant.

The angle *DEF*, Fig. 114 (*b*), is in the second quadrant.

The angle *FED* is in the third quadrant.

Placing these angles so that their initial lines are in the same direction, and dropping perpendiculars to the initial line, as in Fig. 115 (I), (II), (III) and (IV), where the angles *STV* are represented in the four quadrants, we call the perpendicular measured *away* from the initial line the

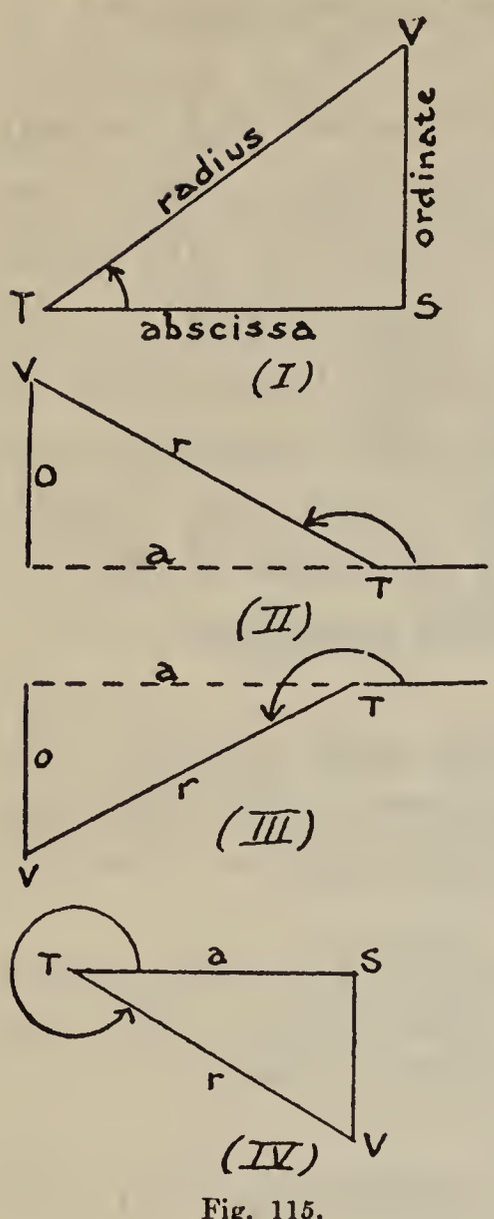

Fig. 115.

ordinate, and designate it by *o*. We call the part of the initial line measured *away* from the vertex to the foot of the perpendicular the *abscissa*, and designate it by *a*. The hypotenuse of the right triangle formed is called the *radius*, on account of its movement in generating the angle, and is designated by *r*. The radius is not thought of as having opposition in direction, and has no sign. The ordinate has an opposite direction in the third and fourth quadrants from that of the first quadrant. Hence we say the ordinate is plus in the first and second quadrants, and minus in the third and fourth quadrants. In like manner, the *abscissa* is plus in the first and fourth quadrants, and minus in the second and third.

4. Trigonometric Functions:—From the ratios of these three lines—radius, abscissa and ordinate—three pairs of elementary trigonometric functions can be made. They are (1) *sine* and *cosine*, (2) *tangent* and *cotangent*, (3) *secant* and *cosecant*. The ratios are as follows:

$$\text{sine } \theta = \frac{\text{ordinate}}{\text{radius}}, \text{ generally written } \sin\theta = \frac{o}{r}$$

$$\text{cosine } \theta = \frac{\text{abscissa}}{\text{radius}}, \text{ generally written } \cos\theta = \frac{a}{r}$$

$$\text{tangent } \theta = \frac{\text{ordinate}}{\text{abscissa}}, \text{ generally written } \tan\theta = \frac{o}{a}$$

cotangent θ = $\frac{\text{abscissa}}{\text{ordinate}}$, generally written cot $\theta = \frac{a}{o}$

secant θ = $\frac{\text{radius}}{\text{abscissa}}$, generally written sec $\theta = \frac{r}{a}$

cosecant θ = $\frac{\text{radius}}{\text{ordinate}}$, generally written csc $\theta = \frac{r}{o}$

5. Algebraic Signs of Functions:—As the radius has no sign (+ or −) the algebraic sign of the sine and cosecant is determined by the ordinate; that of the cosine and secant by the abscissa, and that of the tangent and cotangent by both ordinate and abscissa. The following is evident:

	I	II	III	IV	(Quadrants)
sin θ	+	+	−	−	
cos θ	+	−	−	+	
tan θ	+	−	+	−	
cot θ	+	−	+	−	
sec θ	+	−	−	+	
csc θ	+	+	−	−	

6. Seven Fundamental Formulas.—Between the foregoing six functions there are seven fundamental relationships, which should be carefully noted.

(1) $\sin\theta \times \csc\theta = 1$, since $\frac{o}{r} \times \frac{r}{o} = 1$

(2) $\cos\theta \times \sec\theta = 1$, since $\frac{a}{r} \times \frac{r}{a} = 1$

(3) $\tan\theta \times \cot\theta = 1$, since $\frac{o}{a} \times \frac{a}{o} = 1$

(4) $\tan\theta = \frac{\sin\theta}{\cos\theta}$, since $\frac{o}{a} = \frac{\frac{o}{r}}{\frac{a}{r}}$

Recalling the Pythagorean theorem, we know that in all such triangles,

$$o^2 + a^2 = r^2$$

Dividing this thru by r^2, we have $\frac{o^2}{r^2} + \frac{a^2}{r^2} = 1$.

(5) $\text{Sin}^2\,\theta + \cos^2\theta = 1$.

Dividing thru by a^2, we have $\frac{o^2}{a^2} + 1 = \frac{r^2}{a^2}$.

(6) $\text{Tan}^2\,\theta + 1 = \sec^2\theta$

Dividing thru by o^2, we have $1 + \frac{a^2}{o^2} = \frac{r^2}{o^2}$.

(7) $1 + \cot^2\theta = \csc^2\theta$.

Of the six functions sin θ and cos θ are regarded as the simplest.

A thoro acquaintance with these formulas, not only in the way they are given here, but in their various transformations, is necessary to a clear understanding of what follows.

From the first of the foregoing relationships we get $\csc\theta = \frac{1}{\sin\theta}$. This means that a *csc* θ in the numerator can be replaced by a *sin* θ in the denominator and vice versa.

From another we learn that $\cos\theta = \frac{1}{\sec\theta}$. This means that a *cos* θ in the numerator can be replaced by a *sec* θ in the denominator and vice versa.

From another it is apparent that $\cos\theta = \sqrt{1 - \sin^2\theta}$, since $\cos^2\theta = 1 - \sin^2\theta$.

From still another that $\sec^2\theta - \tan^2\theta = 1$.

Let us now by means of these formulas attempt to show that

(Example 1) $\frac{\sec^2\theta + \csc^2\theta}{\sec\theta\,\csc\theta} = \tan\theta + \cot\theta$.

We have long since learned that $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

Applying this formula, canceling and substituting as in the foregoing, we have:

$$\frac{\sec^2\theta}{\sec\theta\csc\theta}+\frac{\csc^2\theta}{\sec\theta\csc\theta}=\frac{\sec\theta}{\csc\theta}+\frac{\csc\theta}{\sec\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\tan\theta+\cot\theta$$

Since $\sec\theta=\frac{1}{\cos\theta}$, $\frac{1}{\csc\theta}=\sin\theta$, $\csc\theta=\frac{1}{\sin\theta}$, $\frac{1}{\sec\theta}=\cos\theta$, $\frac{\sin\theta}{\cos\theta}=\tan\theta$.

It is always well to reduce a complicated expression to one less complicated, simplifying as much as necessary. *But do not clear of fractions across the equal mark.*

(Example 2) $\frac{1}{\tan^2\theta+1}+\frac{1}{\cot^2\theta+1}=1.$

By consulting our seven fundamental formulas, we see that

$$\frac{1}{\tan^2\theta+1}+\frac{1}{\cot^2\theta+1}=\frac{1}{\sec^2\theta}+\frac{1}{\csc^2\theta}=\cos^2\theta+\sin^2\theta=1$$

(Example 3) $\frac{1-2\cos^2\theta}{\sin\theta\cos\theta}=\tan\theta-\cot\theta.$ By reading our seven fundamental formulas backward, we see $1=\sin^2\theta+\cos^2\theta$; hence

$$\frac{1-2\cos^2\theta}{\sin\theta\cos\theta}=\frac{\sin^2\theta+\cos^2\theta-2\cos^2\theta}{\sin\theta\cos\theta}=\frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}\text{ (simplifying)}$$

$$=\frac{\sin^2\theta}{\sin\theta\cos\theta}-\frac{\cos^2\theta}{\sin\theta\cos\theta}=\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}=\tan\theta-\cot\theta.$$

(Example 4) $\frac{\sin\theta+\tan\theta}{\cot\theta+\csc\theta}=\sin\theta\tan\theta.$

Here again our fundamental formulas show us that

$$\frac{\sin\theta+\tan\theta}{\cot\theta+\csc\theta}=\frac{\sin\theta+\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}+\frac{1}{\sin\theta}}=\frac{\sin\theta\left(1+\frac{1}{\cos\theta}\right)}{\left(\frac{\cos\theta+1}{\sin\theta}\right)}=\frac{\sin^2\theta}{\cos\theta}=\tan\theta\sin\theta$$

These functions are numbers and may therefore be simplified as in arithmetic and algebra. Thus, $a+\frac{a}{b}=a\left(1+\frac{1}{b}\right)$.

For the same reason $\sin\theta+\frac{\sin\theta}{\cos\theta}=\sin\theta\left(1+\frac{1}{\cos\theta}\right)$, $\frac{\frac{\tan\theta}{1}}{\sin\theta}=$

$\tan\theta\sin\theta$ $\left(\text{since }\frac{\frac{a}{1}}{b}=a\times\frac{b}{1}=ab\right)$

(Example 5) $1+\frac{\tan^2}{1+\sec\theta}=\sec\theta.$ (Read fundamental formulas.)

For $1 + \frac{\tan^2\theta}{1+\sec\theta} = 1 + \frac{\sec^2\theta - 1}{\sec\theta + 1} = 1 + (\sec\theta - 1) = \sec\theta$ as $\sec^2\theta - 1 = (\sec\theta + 1)(\sec\theta - 1)$.

7. Numerical Values of the Functions.—It must always be remembered that—as has been said—these functions are numbers. Let us take for example, the right triangle, Fig. 116, whose sides are 3, 4 and 5. Here, according to our definitions of functions, sin A is $\frac{3}{5}$, cos A is $\frac{4}{5}$, tan A is $\frac{3}{4}$, cot A is $\frac{4}{3}$, sec A is $\frac{5}{4}$, csc A is $\frac{5}{3}$.

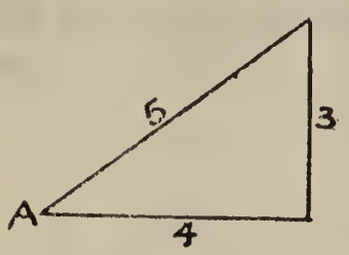

Fig. 116.

If we divide a square, Fig. 117, by a diagonal we will get two isosceles right triangles. We can represent the sides of one of these triangles as 1, 1, $\sqrt{2}$. Then sin 45° = $\frac{1}{\sqrt{2}}$ = cos 45°; tan 45° = 1 = cot 45°; sec 45° = $\sqrt{2}$ = csc 45°.

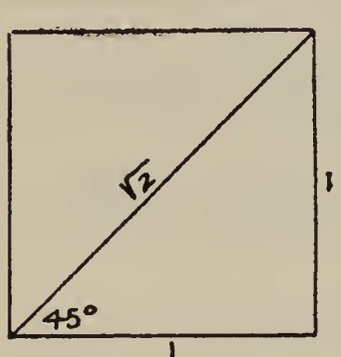

Fig. 117.

We can form a 30° 60° right triangle from the equilateral triangle, Fig. 118, and represent its sides by 1, 2 and $\sqrt{3}$. Sin 30° = $\frac{1}{2}$; cos 30° = $\frac{\sqrt{3}}{2}$; tan 30° = $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$; cot 30° = $\sqrt{3}$; sec 30° = $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$; csc 30° = 2; or, in decimals, sin 30° = 0.5000, cos 30° = 0.8660, tan 30° = 0.5774, cot 30° = 1.732, sec 30° = 1.155, csc 30° = 2.000.

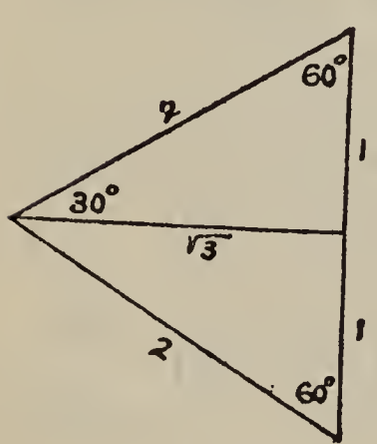

Fig. 118.

It will be well to check the foregoing with the seven fundamental formulas. Ex. $\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$, etc.

8. Angles Greater Than 90°.—Trigonometric tables are computed only for angles up to 45°. We can obtain the functions of any angle in terms of angles less than 90° or, if desired, less than 45°. In Fig. 119, the

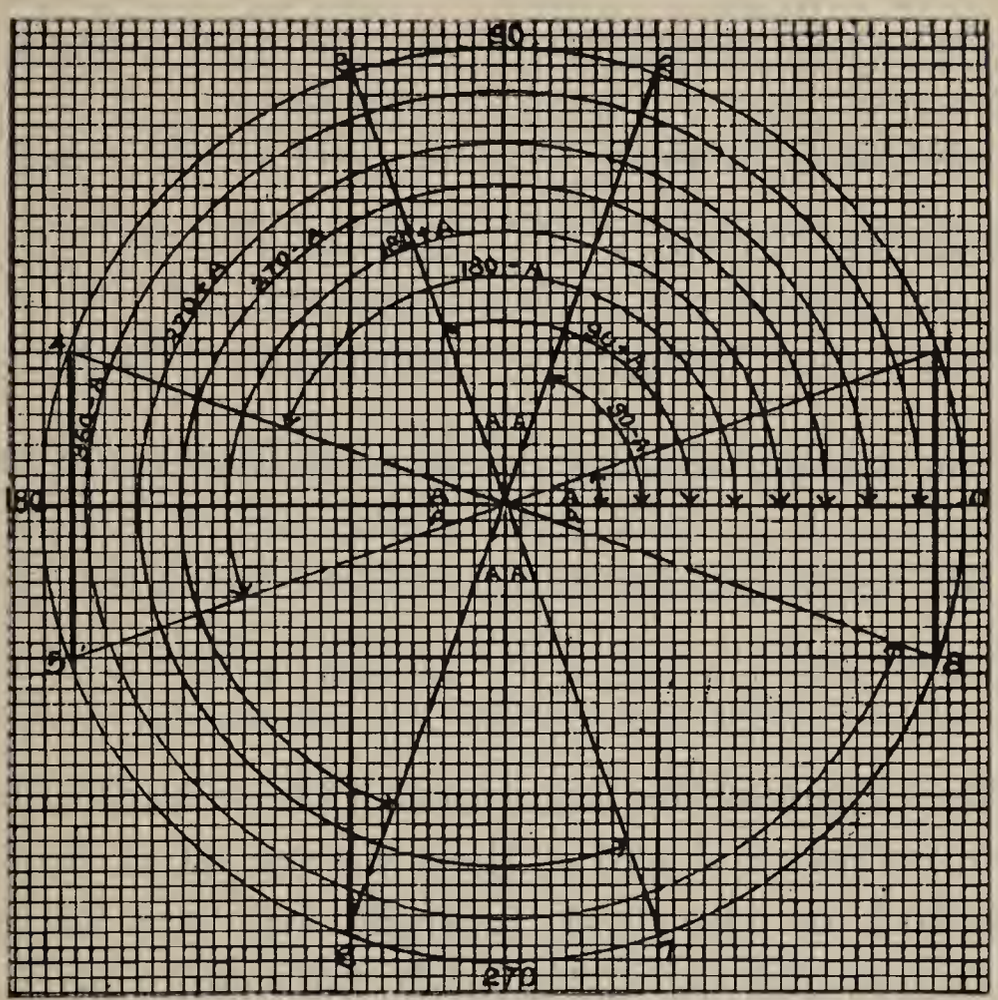

Fig. 119.

eight right triangles whose hypotenuses are the radii which cut the circle at 1, 2, 3, 4, 5, 6, 7 and 8 are obviously equal. Both the hypotenuse and the angle A are equal. The angles at 2, 3, 6, 7 are equal to the angle A, since alternate interior angles of parallel lines are equal. We will denote the equal sides of each of these eight equal triangles by r = hypotenuse; l = long side and s = short side. We will also discuss the sign of angles in the different quadrants and generalize. Let us remember that the sine of an angle equals $\frac{\text{ordinate}}{\text{radius}}$ and that the cotangent equals the

$\frac{\text{abscissa}}{\text{ordinate}}$. The reader should make tables for cos, tan, sec and csc. The angles are marked $(90° - A)$, $(90° + A)$, etc. We now wish to express them in terms of angle A in the first triangle.

I. $\sin\ (90°-A)=\frac{l}{r}=\cos A$	$\cot\ (90-A)=\frac{s}{l}=\tan A$
II. $\sin\ (90°+A)=\frac{l}{r}=\cos A$	$\cot\ (90+A)=\frac{-s}{l}=-\tan A$
$\sin\ (180°-A)=\frac{s}{r}=\sin A$	$\cot\ (180-A)=\frac{-l}{s}=-\cot A$
III. $\sin\ (180°+A)=\frac{-s}{r}=-\sin A$	$\cot\ (180+A)=\frac{-l}{-s}=\frac{l}{s}=\cot A$
$\sin\ (270°-A)=\frac{-l}{r}=-\cos A$	$\cot\ (270-A)=\frac{-s}{-l}=\frac{s}{l}=\tan A$
IV. $\sin\ (270°+A)=\frac{-l}{r}=-\cos A$	$\cot\ (270+A)=\frac{s}{-l}=-\frac{s}{l}=-\tan A$
$\sin\ (360°-A)=\frac{-s}{r}=-\sin A$	$\cot\ (360-A)=\frac{-l}{s}=-\frac{l}{s}=-\cot A$

We first express the functions of these angles in terms of the numbers r, l and s, then we see what function is represented in the first triangle by our ratio. We must be careful to distinguish between positive and negative ordinates and abscissas. It will be noticed that the signs are settled by the quadrants, as in Section 5 of this Chapter. Also that when 90 or 270 (an odd number of times 90) is used, the function changes to the co-function, but that when 180 or 360 (an even number of times 90) is used, the same function occurs, that is, $\sin (90 - A) = \cos A$; but $\sin (180 - A) = \sin A$.

Also that $\tan (270 - A) = \cot A$, and that $\tan (180 - A) = -\tan A$. The rule is: *first settle the sign, then settle the function.* $\sin (75°) = \sin (90° - 15°) = +\cos 15°$. $\sin (100°) = \sin (90° + 10) = +\cos 10°$. $\tan 150° = \tan (180° - 30°) = -\tan 30°$. $\cot (200°) = \cot (180° + 20°) = +\cot 20°$. $\sec (250°) =$

sec $(270° - 20°) = -\csc 20°$. Csc $300° = \csc (270° + 30°) = -\sec 30°$. Sin $100° 10' = \sin (90° + 10° 10') = \cos 10° 10'$. Csc $260° 13' 13'' = \csc (180° + 80° 13' 13'') = -\csc 80° 13' 13''$. It is very desirable to become expert in this operation.

Evidently since $\sin 0° = 0$ and $\cos 0° = 1$, it follows that $\sin 90° = \sin (90° + 0°) = \cos 0° = 1$. Similarly, $\sin (180°) = \sin (180° + 0) = -\sin 0° = 0$. Again, $\sin 270° = \sin (270° + 0°) = -\cos 0° = -1$. Similarly $\cos 90° = \cos 270° = 0$. Cos $180° = -1$. Tan $0° = 0$. Tan $90° = \tan (90° + 0) = \cot 0° = \frac{1}{\tan 0°} =$ infinity, where infinity means that as the angle gets nearer to 90°, the tangent increases indefinitely. Note that whereas $\tan 180° = 0$, $\tan 270° = -$ infinity.

9. Inverse Functions.—From the foregoing we learn that $\sin 30° = \sin 150° = \sin 390° = \sin 510°$, etc. Inversely we say the angle whose sine is one-half is 30°, 150°, 390°, 510°, etc. We abbreviate this $\sin^{-1} \frac{1}{2}$ or arc sin $\frac{1}{2}$, which are read "angle whose sine is one-half" and "arcsin $\frac{1}{2}$". One of the values of $\sin^{-1} \frac{1}{\sqrt{2}} + \cos^{-1} \frac{1}{\sqrt{2}}$ is 90°.

The expression $\sin (\cos^{-1} \frac{1}{2})$ also means: given the cosine of an angle to be $\frac{1}{2}$, find its sine. Sin $\theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \pm \frac{1}{2}\sqrt{3}$. Remember arcsin a, $\tan^{-1} a$ are angles.

10. Value of $\frac{\sin \theta}{\theta}$.—Referring to Fig. 120 we see that

$$2 \sin \theta < 2\theta < 2 \tan \theta$$

$$\text{Hence } 1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

$$\text{therefore } 1 > \frac{\sin \theta}{\theta} > \cos \theta$$

As $\cos 0° = 1$, $= \frac{\sin \theta}{\theta}$ approaches 1 as θ approaches 0.

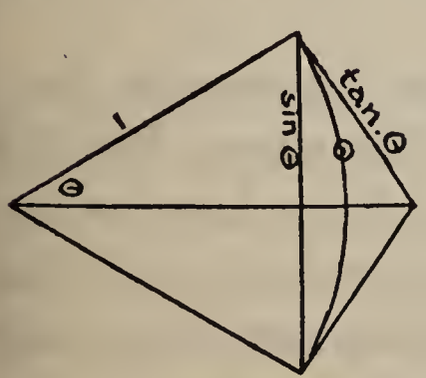

Fig. 120.

11. The Right Triangle.—We will now take the right triangle ABC, Fig. 121, placing C at the vertex of the right angle and lettering the sides opposite the angles a, b, c. Here

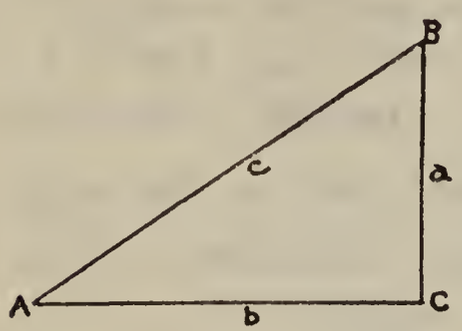

Fig. 121.

$$\sin A = \frac{a}{c} = \cos B$$
$$\cos A = \frac{b}{c} = \sin B$$
$$\tan A = \frac{a}{b} = \cot B$$
$$\cot A = \frac{b}{a} = \tan B$$
$$\sec A = \frac{c}{b} = \csc B$$
$$\csc A = \frac{c}{a} = \sec B$$

Note that for functions of angle A the ordinate is a, while for functions of angle B the ordinate is b.

These equations show us that each function of the angle B (which is complementary to the angle A) is the *co-function* of the angle A. This means that sin 30° = cos 60°; cot 42° = tan 48°; sec 35° 15′ = csc 54° 45′; sin 100° = cos (− 10°).

We can also obtain from these equations nine different expressions for a, as follows: $a = c \sin A = c \cos B = b \tan A = b \cot B = \frac{b}{\cot A} = \frac{b}{\tan B} = \frac{c}{\csc A} = \frac{c}{\sec B} = \sqrt{(c-b)(c+b)}$.

$a = \sqrt{c^2 - b^2}$, from $a^2 + b^2 = c^2$; also $c^2 - b^2 = (c-b)(c+b)$.

Nine expressions for b, and nine expressions for c can be found in the same manner. It is very desirable that the reader should do this, and learn how to read with facility a side of a triangle in terms of two other parts. This is the alphabet of the subject, and must be mastered if further progress is to be made.

12. Solution of the Right Triangle.—The solution of any right triangle, as, for instance BCA, Fig. 121, can be performed as follows:

Given:	To find:
$A = 31$	$B = 59°$
$a = 16.235$	$b = 27{,}020$
	$c = 31{,}522$

Since $A + B = 90°$, $B = 59°$. We may now write formulas for b and c in terms of the *given* parts A and a, and also a check formula in terms of both *known* and *unknown* parts.

$b = a \cot A$

$1.6643 = \cot A$
$16.235 = a$
16.643
9.986
333
50
8
27.020 = b

$c = \frac{a}{\sin A}$

$\sin A = .51504)\overline{16.235} = a$

31.52 = c
15.451
784
515
269
258
11
10

Check: $b = \sqrt{(c-a)(c+a)}$

$c = 31.52$
$a = 16.24$
$c - a = 15.28$
$c + a = 47.76$
611.2
107.0
10.7
9

729.8 | 27.02
4 | 4
329 |
329 | 47
8 | 54

We were given $A = 31°$ and $a = 16.235$ and will mark the given parts on Fig. 121, finding each new part in terms of the given parts. Note that there is always an error in computation with inexact numbers, and if we use a computed part in a new computation, there will be additional error. We should therefore, always attempt to use original data.

By referring to the table of values (pages 186–190) we find that cot 31° = 1.6643, and sin 31° = .51504. Performing the indicated operations, we get $b = 27.020$ and $c = 31.52$; $B = 59°$, the complement of A. We check this computation by getting b in another way, from other parts of the triangle. The formula $b = \sqrt{(c - a)(c + a)}$ or one similar to it should always appear in every right-triangle computation. We multiply $(c - a)$ by $(c + a)$ and then take the square root of this product. The first three figures of the square root are 27.0. As we have three figures we can obtain the next one by long division, as we did when we were studying square root. Now 54 goes into 80 nearly 2 times. As the two values of b agree, we are reasonably certain of the accuracy of our results.

13. Solution by Logarithms.—Now solving this same problem by means of logarithms and using the same formulas:

$$b = a \cot A \qquad c = \frac{a}{\sin A} \qquad b = \sqrt{(c - a)(c + a)}$$

We draw up a skeleton form, such as the following:

$\log a =$ 1.	$\log a =$ 1.	$\log (c-a) =$
$\log \cot A =$	$\operatorname{colog} \sin A =$	$\log (c+a) =$
$\log b =$	$\log c =$	2)
$b =$	$c =$	$\log b =$
	$a = 16.235$	
	$c-a =$	
	$c+a =$	

We will now repeat this skeleton, and fill out the blanks.

$\log a = 1.21046$	$\log a = 1.21046$	$\log (c-a) = 1.18432$
$\log \cot A = 0.22123$	$\operatorname{colog} \sin A = 0.28816$	$\log (c+a) = 1.67903$
$\log b = 1.43169$	$\log c = 1.49862$	2) 2.86335
$b = 27.020$	$c = 31.522$	$\log b = 1.43168$
	$a = 16.235$	
	$c-a = 15.287$	
	$c+a = 47.757$	

14. Exercises on Right Triangles.—In Fig. 122 we have another right triangle BCA, some of the parts of which are given and others are to be found.

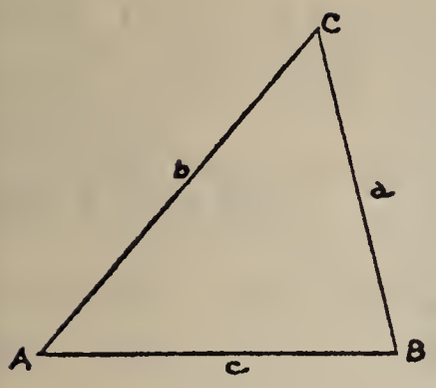

Fig. 122.

Given:	To find:
$c = 62.134$	$a =$
$b = 41.275$	$A =$
$c - b = 20.859$	$B =$
$c + b = 103.409 = 103.41$	

$a = \sqrt{(c-b)(c+b)}$ $\quad \cos A = \frac{b}{c}$ $\quad$ Check $\tan A = \frac{a}{b}$

$\log (c-b) = 1.31929$	$\log b = 1.61569$	$\log a = 1.66692$
$\log (c+b) = 2.01456$	$\operatorname{colog} c = 8.20667$	$\operatorname{colog} b = 8.38431$
$2)\overline{3.33385}$	$\log \cos = A = 9.82236$	$\log \tan A = 0.05123$
$\log a = 1.66692$		
$a = 46.640$	$A = 48°22'17''$	$A = 48°22'10''$

Looking up logs of $c - b$, $c + b$, b and c, we note that colog b can be written immediately from log b. Care is needed in getting the angle from log cos A, as the mantissas are decreasing as the angle increases. The difference of $7''$ in the check is well within the limit of accuracy. No pencil need be used except in writing the figures as seen here. The tangent is the best function to check with, if the form $a = \sqrt{(c-b)(c-b)}$ has been used previously in the computation.

The following is still another solution of the right triangle, as represented by Fig. 122.

Given:	To Find:
$a = 163.89$	$A = 48°\ 57'\ 35''$
$b = 142.67$	$B = 41°\ 02'\ 25''$
	$c = 217.29$

Here we have no direct logarithmic formula for c.

$\tan A = \frac{a}{b}$ | $c = \frac{a}{\sin A}$ | $a = \sqrt{(c-b)(c+b)}$

$\log a = 2.21455$	$\log a = 2.21455$	$\log (c-b) = 1.87286$
$\operatorname{colog} b = 7.84567$	$\operatorname{colog} \sin A = 0.12249$	$\log (c+b) = 2.55625$
$\log \tan A = 0.06022$	$\log c = 2.33704$	$2)4.42911$
	$c = 217.29$	$\log a = 2.21456$
	$b = 142.67$	
$A = 48° 57' 35''$	$c-b = 74.62$	
	$c+b = 359.96$	

To subtract 48° 57′ 35″ from 90° to get the complement B note that 90° is 89° 59′ 60″. All the work done is before us. There is no scratch-work of any kind. In getting the cologs, the characteristics were taken from 9 before looking into the table, then the numbers which were alike in the two logarithms were subtracted from 9, so that only small numbers, generally of the last two places, had to be adjusted after interpolation.

15. Oblique Triangles.—Oblique triangles, having three parts given, one of which is a side, may be worked by dividing them into right triangles, tho the case when three sides are given presents some difficulties. Generally, these problems are solved by other formulas than the simple ones just given. We will now deduce the formulas needed and in doing so will obtain a number of other useful expressions, involving sums, differences, multiples and fractions of angles.

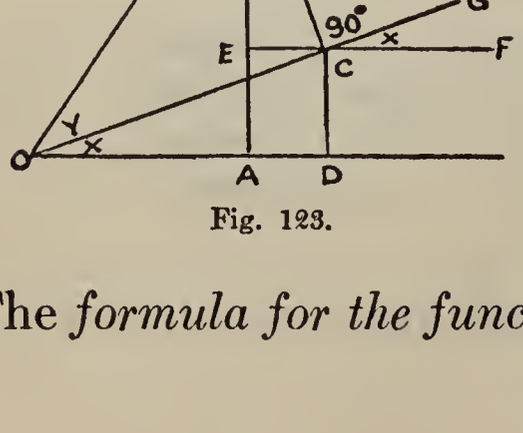

Fig. 123.

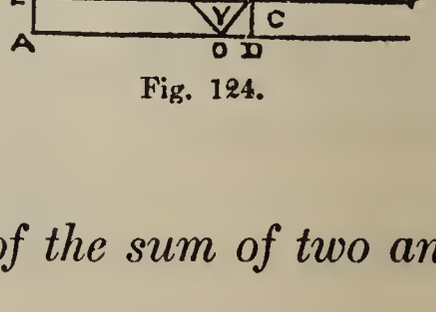

Fig. 124.

The *formula for the functions of the sum of two angles*

is deduced as follows, Figs. 123 and 124 being lettered alike:

The angle *DOB* is the sum of the angles x and y. In Fig. 123 both are acute, as is their sum, while in Fig. 124 both angles are acute but their sum is obtuse. The angle *FCB*, in both figures, is $90° + x$. (*Ordinates* are read from the *foot* of the perpendicular, *abscissas* from the *vertex* of the angle.)

$$\sin (x + y) = \frac{AB}{OB} = \frac{AE+EB}{OB} = \frac{DC+EB}{OB} = \frac{DC}{OB} + \frac{EB}{OB}$$

(We have chosen to express it in terms of the sides of the right triangles containing the angles x and y.)

$$\frac{DC}{OB} = \frac{DC}{OC} \times \frac{OC}{OB} = \sin x \cos y.$$

(*OC* is common to the two triangles which contain *DC* and *OC*. Multiplying both numerator and denominator by the same number does not, as we know, change the value.)

Likewise:

$$\frac{EB}{OB} = \frac{EB}{CB} \times \frac{CB}{OB} = \sin (90° + x) \sin y = \cos x \sin y.$$

(*CB* is common to the triangles which contain *EB* and *OB*.)

Therefore $\sin (x + y) = \sin x \cos y + \cos x \sin y$

In a like manner $\cos (x + y) = \frac{OA}{OB} = \frac{OD+DA}{OB} = \frac{OD+CE}{OB}$

$= \frac{OD}{OB} + \frac{CE}{OB}$ $\qquad \frac{OD}{OB} = \frac{OD}{OC} \times \frac{OC}{OB} = \cos x \cos y$

$$\frac{CE}{OB} = \frac{CE}{CB} \times \frac{CB}{OB} = \cos (90° + x) \sin y = -\sin x \sin y.$$

Therefore $\cos (x + y) = \cos x \cos y = -\sin x \sin y.$

16. Functions of Sum of Two Angles.—The preceding deduction proves for all cases in which the angles to be added are acute that *the sine of the sum of two*

angles is the sine of the first multiplied by the cosine of the second, plus the cosine of the first multiplied by the sine of the second, and that *the cosine of the sum of two angles equals the product of their cosines minus the product of their sines.*

The form of the proof is perfectly general, whether the angles x and y are acute or obtuse.

As an example we will show that since $\sin 90° = \sin (60° + 30°) = \sin 60° \cos 30° + \cos 60° \sin 30° = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1.$

We are confirmed in our former deduction that $\sin 90° = 1$, Also that $\cos 225° = \cos (180° + 45°) = \cos 180° \cos 45° - \sin 180° \sin 45° = -1 \times \frac{\sqrt{2}}{2} - 0 = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$, which we would have known before since $\cos 225° = \cos (180 + 45) = \cos 45° = -\frac{1}{\sqrt{2}}$.

As $\tan A = \frac{\sin A}{\cos A}$, then $\tan (x + y) =$

$$\frac{\sin (x+y)}{\cos (x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

In the foregoing we divided each term by $\cos x \cos y$ in order to get our result in terms of the tangent. In like manner, by dividing thru the expansion of

$$\cot (x + y) = \frac{\cos (x+y)}{\sin (x+y)}$$

by $\sin x \sin y$, we get $\cot (x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$

Similar formulas for the functions of $x - y$ may be written by substituting $(-y)$ for (y) and remembering $\sin (-y) = \sin (360° - y) = -\sin y$, $\cos (-y) = \cos (360° - y) = \cos y$, $\tan (-y) = \tan (360° - y) = -\tan y$, $\cot (-y) = \cot (360° - y) = -\cot y$, etc.

The two sets of formulas may be combined as follows:

$$\sin\ (x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos\ (x \pm y) = \cos x \cos y \pm \sin x \sin y$$

$$\tan\ (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\cot\ (x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$$

(Read either all upper signs or all lower signs.)

17. Functions of Double Angle and Half Angle.—The formulas for $2x$ can be obtained by remembering

$$\sin 2x = \sin\ (x + x) = \sin x \cos x + \cos x \sin x$$

$$= 2 \sin x \cos x, \text{ since } 2x = x + x$$

$$\cos 2x = \cos\ (x + x) = \cos x \cos x - \sin x \sin x$$

$$= \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$

Since $\sin^2 x + \cos^2 x = 1$

$$\text{Likewise } \tan 2x = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\cot 2x = \frac{\cot x \cot x - 1}{\cot x + \cot x} = \frac{\cot^2 x - 1}{2 \cot x}$$

These formulas should be thoroughly digested and used under all sorts of conditions. For instance, $\sin 2x = 2 \sin x \cos x$ enables us to write $\sin 6x = 2 \sin 3x \cos 3x$, since $6x$ is twice $3x$. Also $\cos 2x = 1 - 2 \sin^2 x$ enables us to write $\cos x = 1 - 2 \sin^2 \frac{1}{2} x$, since x is twice $\frac{1}{2} x$. This last formula enables us to write $2 \sin^2 \frac{1}{2} x = 1 - \cos x$, whence $\sin^2 \frac{1}{2} x = \frac{1 - \cos x}{2}$, a most useful formula in trigonometry and calculus. Similarly, $\sin^2 x = \frac{1 - \cos 2x}{2}$, also $\sin^2 2x = \frac{1 - \cos 4x}{2}$.

From the formula $\cos 2x = 2 \cos^2 x - 1$, we easily obtain $\cos^2 \left(\frac{1}{2} x\right) = \frac{1 + \cos x}{2}$

Since $\tan^2 \frac{1}{2}x = \frac{\sin^2 \frac{1}{2}x}{\cos^2 \frac{1}{2}x} = \frac{1-\cos x}{1+\cos x} = \frac{(1-\cos x)^2}{1-\cos^2 x}$ or $= \frac{1-\cos^2 x}{(1+\cos x)^2}$

we have $\tan \frac{1}{2}x = \frac{1-\cos x}{\sin x}$ or $\frac{\sin x}{1+\cos x}$ $\quad (1-\cos^2 x = \sin^2 x)$

$\cot \frac{1}{2}x = \frac{\sin x}{1-\cos x}$ or $\frac{1+\cos x}{\sin x}$ $\quad$ (tangent = cotangent)

18. Sums and Differences of Functions.—Another necessary and useful set of formulas is obtained as follows:

$\sin (x+y) = \sin x \cos +y \cos x \sin y$
$\sin (x-y) = \sin x \cos y - \cos x \sin y$

$\sin (x+y) + \sin (x-y) = 2 \sin x \cos y$ (Adding)
$\sin (x+y) - \sin (x-y) = 2 \cos x \sin y$ (Subtracting)

Call $x+y=A$
$x-y=B$

Then $x = \frac{A+B}{2}$, $y = \frac{A-B}{2}$, and the formula

$\sin (x+y) + \sin (x-y) = 2 \sin x \cos y$ becomes

$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$

Likewise, $\sin (x+y) - \sin (x-y) = 2 \cos x \sin y$, becomes $\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$

Similarly, from

$\cos (x+y) + \cos (x-y) = 2 \cos x \cos y$, and

$\cos (x+y) - \cos (x-y) = -2 \sin x \sin y$, we have

$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$, and

$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$

An example of the use of these formulas:

$\sin 150° + \sin 90° = 2 \sin 120° \cos 30°$

(Since $\frac{150+90}{2} = 120$, and $\frac{150-90}{2} = 30$)

This gives the identity $\frac{1}{2} + 1 = \left(2 \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2}\right) = \frac{3}{2}$.

19. The Law of Sines.—In the triangles ABC, Figs. 125 and 126, $\sin B = \frac{p}{a}$ and $\sin A = \frac{p}{b}$.

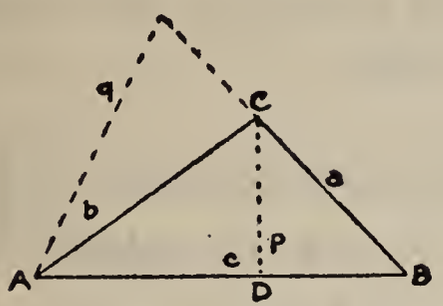

Fig. 125.

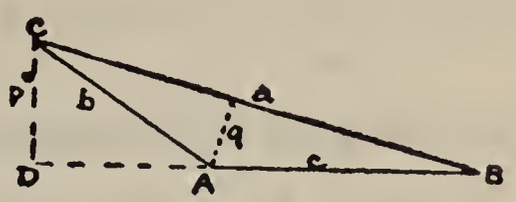

Fig. 126.

Dividing, we have $\frac{\sin A}{\sin B} = \frac{a}{b}$, whence $\frac{a}{\sin A} = \frac{b}{\sin B}$

Similarly, drawing the perpendicular q from A to the side a, we have

$\sin B = \frac{q}{c}$ and $\sin C = \frac{q}{b}$; Dividing we have $\frac{\sin B}{\sin C} = \frac{b}{c}$,

whence $\frac{b}{\sin B} = \frac{c}{\sin C}$. Combining the two, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

20. Numerical Solution of Oblique Triangle by Law of Sines.—We will solve a problem by the foregoing formula; first referring to Fig. 127.

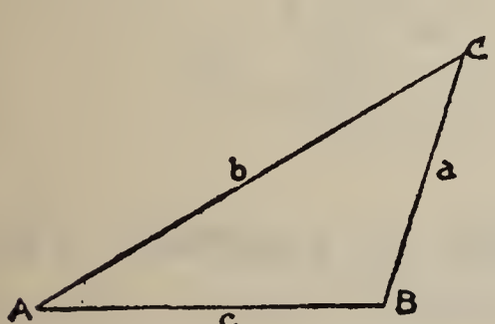

Fig. 127.

Given:	To find:
$a = 628.19$	$C = 39°\,04'\,12''$
$A = 68°\,39'\,09''$	$b = 642.47$
$B = 72°\,16'\,39''$	$c = 425.10$

$$b = \frac{a \sin B}{\sin A} \qquad c = \frac{a \sin C}{\sin A}$$

$\log a = 2.79809$	$\log a = 2.79809$
$\log \sin B = 9.97889$	$\log \sin C = 9.79953$
$\operatorname{colog} \sin A = 0.03087$	$\operatorname{colog} \sin A = 0.03087$
$\log b = 2.80785$	$\log c = 2.62849$
$b = 642.47$	$b = 425.10$

There is no check by this formula.

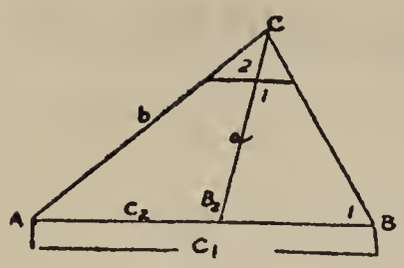

Fig. 128.

When two sides and the angle opposite one of them are given there are two solutions, as in Fig. 128, unless ruled out by the principle that the greater side is opposite the greater angle.

21. The Law of Tangents.—From the formula $\frac{\sin A}{\sin B} = \frac{a}{b}$, (Section 19 of this chapter) we get, by adding 1 to each side, and reducing to common denominator,

$$\frac{\sin A}{\sin B} + 1 = \frac{a}{b} + 1; \quad \frac{\sin A + \sin B}{\sin B} = \frac{a+b}{b}$$

and by subtracting 1, $\frac{\sin A - \sin B}{\sin B} = \frac{a-b}{b}$

Dividing these two equations,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$$

$$\frac{a-b}{a+b} = \cot\frac{A+B}{2}\tan\frac{A-B}{2} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}$$

Other forms of this formula are:

$$\frac{b-c}{b+c} = \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}; \quad \frac{c-a}{c+a} = \frac{\tan\frac{C-A}{2}}{\tan\frac{C+A}{2}}, \text{ etc.}$$

22. Numerical Solution by Law of Tangents.—The formula of the law of tangents enables us to solve a triangle when two sides and the included angle are given. If we have the parts a, c, and B in Fig. 129, we can subtract B from 180° and get $A + C$, and hence $\frac{A+C}{2}$. In the formula

$$\frac{c-a}{c+a} = \frac{\tan\frac{C-A}{2}}{\tan\frac{C+A}{2}}$$ we can obtain $\tan\frac{C-A}{2}$ from the

three known parts and then get $\frac{C-A}{2}$ from the table. Adding $\frac{C-A}{2}$ to $\frac{C+A}{2}$ gives us C. Subtracting, we obtain A. We now know all the parts of the triangle except the side b, which can be obtained from the sine formula in two ways, giving us a check.

We will solve the following problem by the use of this formula.

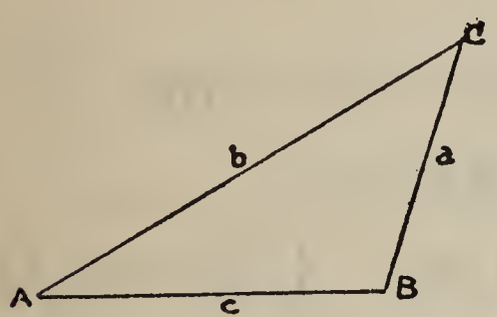

Fig. 129.

Given:	To Find:
$a=82$	$b=196.31$
$c=167$	$A=24°\ 25'\ 10''$
$B=98°\ 14'$	$C=57°\ 20'\ 50''$
$C+A=81°\ 46'$	

$$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\tan\frac{C+A}{2}$$

$\frac{C+A}{2}=40°\ 53'\ 00''$ $\quad$ $\log\tan=9.93738$

$c-a=85$ $\quad$ $\log=1.92942$

$c+a=249$ $\quad$ $\operatorname{colog}=7.60380$

$\log\tan\frac{C-A}{2}=9.47060$

$\frac{C-A}{2}=16°\ 27'\ 50''$

$\frac{C+A}{2}=40°\ 53'\ 00''$

$C=57°\ 20'\ 50''$

$A=24°\ 25'\ 10''$

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$b=\frac{a\sin B}{\sin A}=\frac{c\sin B}{\sin C}$$

$\log a=1.91381$	$\log c=2.22272$
$\log\sin B=9.99550$	$\log\sin B=9.99550$
$\operatorname{colog}\sin A=0.38364$	$\operatorname{colog}\sin C=0.07471$
$\log b=2.29295$	$\log b=2.29293$
$b=196.30$	

This formula is used in surveying a triangle when we set up the instrument at an angle of the triangle and measure two sides from the instrument with the chain or tape. We can find all the parts with one "setup" of the instrument.

23. Generalized Pythagorean Theorem.—With the same figures as in Section 19 of this chapter we find:

$$\begin{aligned} a^2 &= p^2 + \overline{DB}^2 \\ &= p^2 + (\overline{DA} + c)^2 \qquad (c = AB) \\ &= p^2 + \overline{DA}^2 + 2\overline{DA}c + c^2 \\ a^2 &= b^2 - 2bc\cos A + c^2 \quad \left(\frac{AD}{b} = \cos A\right) \end{aligned}$$

This is a generalized statement of "the square on a hypotenuse," for if $A = 90°$ the formula becomes $a^2 = b^2 + c^2$, since $\cos 90° = 0$. The formula is not, however, adapted to logarithmic work, on account of its having plus and minus signs.

24. Three-Side Formula.—We will use the preceding formula, together with our formula for $\sin\frac{A}{2}$, to develop a formula used by Hero of Alexandria some 2000 years ago. From Section 23, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, hence $2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} =$

$$\frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc}$$

Hero did what was equivalent to calling $(a + b + c) = 2s$. Subtracting $2c$ from both sides of this equation, we get $a + b - c = 2s - 2c = 2(s - c)$. Similarly $a - b + c = 2(s - b)$, and $b + c - a = 2(s - a)$. Canceling out the 2's, our formula becomes

$$\sin^2\frac{A}{2} = \frac{(s - c)(s - b)}{bc}$$

From the formula $2\cos^2\frac{A}{2} = 1 + \cos A$, we may get in a similar manner $2\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$. We may also obtain this formula from the fact that $\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1$.

From these two formulas, by division, we obtain

$$\tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)}$$

This can be written $\tan^2\frac{A}{2} = \frac{(s-a)(s-b)(s-c)}{s(s-a)^2}$

Calling $\frac{(s-a)(s-b)(s-c)}{s} = r^2$. (It can be shown that r is the radius of the inscribed circle.)

$$\tan^2\frac{A}{2} = \frac{r^2}{(s-a)^2},$$

$$\tan\frac{A}{2} = \frac{r}{s-a}$$

This formula enables us to get the half angles of a triangle. If their sum is within 12″ of 90°, using a five-place table of logarithms, we consider them sufficiently accurate, and call their doubles the angles of our triangle.

25. Numerical Solution by Three-Side Formula.—The three-side formula can be used to solve a triangle, as Fig. 130 when the three sides are given.

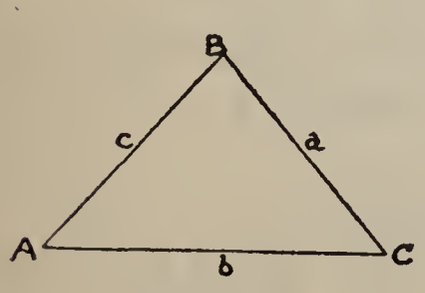

Fig. 130.

Formulas:

$$r = \frac{\sqrt{(s-a)(s-b)(s-c)}}{s}$$

$$\tan\frac{A}{2} = \frac{r}{s-a}$$

$$\tan\frac{B}{2} = \frac{r}{s-b}$$

$$\tan\frac{C}{2} = \frac{r}{s-c}$$

Given:	To Find:
$a = 2.51$	$A = 57^\circ\ 52'\ 48''$
$b = 2.79$	$B = 70^\circ\ 17'\ 28''$
$c = 2.33$	$C = 51^\circ\ 49'\ 52''$
$2s = 7.63$	$A+B+C = 180^\circ\ 00'\ 08''$ (Check)
$s = 3.815$	$\operatorname{colog} s = 9.41851$
$s-a = 1.305$	$\log (s-a) = 0.11561$
$s-b = 1.025$	$\log (s-b) = 0.01072$
$s-c = 1.485$	$\log (s-c) = 0.17173$
$3s-2s = 3.815 = s$	$\log 2r = 19.71657-20$
	$\log r = 9.85829$
	$\log (s-a) = 0.11561$
	$\log \tan \frac{A}{2} = 9.74268$
	$\frac{A}{2} = 28^\circ\ 56'\ 24''$

$\log r = 9.85829$	$\log r = 9.85829$
$\log (s-b) = 0.01072$	$\log (s-c) = 0.17173$
$\log \tan \frac{B}{2} = 9.84757$	$\log \tan \frac{C}{2} = 9.68656$
$\frac{B}{2} = 35^\circ\ 08'\ 44''$	$\frac{C}{2} = 25^\circ\ 54'\ 56''$

This completes the types of triangles that arise in numerical trigonometry. Areas can be obtained by finding altitudes and remembering area = ½ base × altitude. Here altitude on b is c sin A or a sin C.

REVIEW

1. How is the number of radians in an angle found?
2. What is meant by the sine and cosine of an angle? How are these determined when the angle is greater than 90°?
3. What is meant by inverse functions?
4. How could you find the height of a flag-pole?
5. How could you find an angle of a triangle if you know the length of its three sides?

CHAPTER XIII

ALGEBRAIC GEOMETRY

1. Graphical Representation.—If related numbers change so as always to remain in the same ratio, one is said to *vary as* or *vary directly with* the other. "Vary as" is another expression for "is proportional to." Simple interest varies as the time. If y varies with x, $\frac{y}{x} = c$ or $y = cx$. Where c is the constant ratio of y to x, it is sometimes called the *slope* in graphs.

If \$100 is placed on interest at 2% simple interest, the interest, i, varies with the number of years, t. The spaces on the horizontal scale, Fig. 131, represent the number of years; those on the vertical scale, the number of dollars interest. As the interest is 2% or \$2 per hundred we will draw the graph thru the zero mark and a point one space to the right and two spaces up as in the diagram. This graph will pass thru points whose distance up is twice their distance to the right of the origin o.

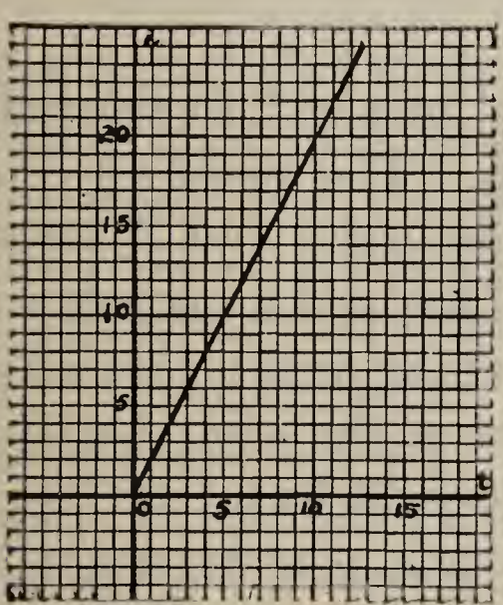

Fig. 131.

To find the interest in 5 years, we look where the line thru 5 on the horizontal scale passes thru the graph and then see what horizontal line passes thru the same point. It is evidently the line thru 10.

How long will it take to get \$6 interest on \$100? By following the reverse course, we find the time

to be 3 years. This graph works very similarly to our multiplication table on page 8 of this volume.

2. Plotting a Graph.—Let us construct the graph

$$3x + 4y = 7.$$

The horizontal scale is x and the vertical scale is y, Fig. 132. Simultaneous values for this expression can be found in Chapter VIII. We will plot the graph of a number of these values, calling x's to the right of zero *plus* and to the left *minus*, also y's upward *plus* and y's downward *minus*. The point $(x) = 1$, $y = 1$, or (1, 1), as we say, means 1 to the right and 1 up. The point (5 − 2) is 5 to the right and 2 down.

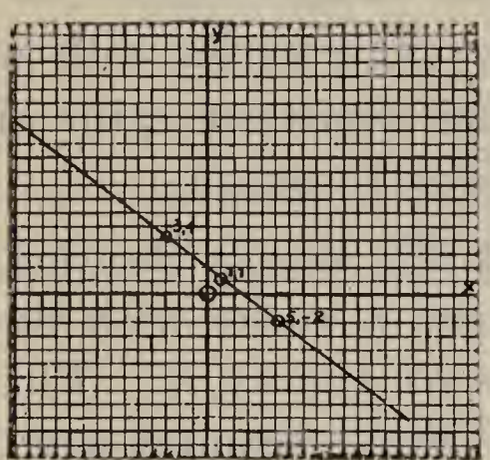

Fig. 132.

The point (− 3, 4) is 3 to the left and 4 up. It will be noticed that a straight line can be drawn thru all of these points and if we try the rest of the points we shall find that they fit on this same straight line.

Now plot $y = 2x$, also $y = 2x + 3$. We notice (Fig. 133) that $y = 2x + 3$ is a straight line parallel to $y = 2x$, with the corresponding points three above. And we further notice that $y = 2x$ is the same line as we plotted in Fig. 131. Also $y = 3$ is a straight line parallel to the $x-$axis three units above it. The equation of the $x-$axis is $y = o$. The

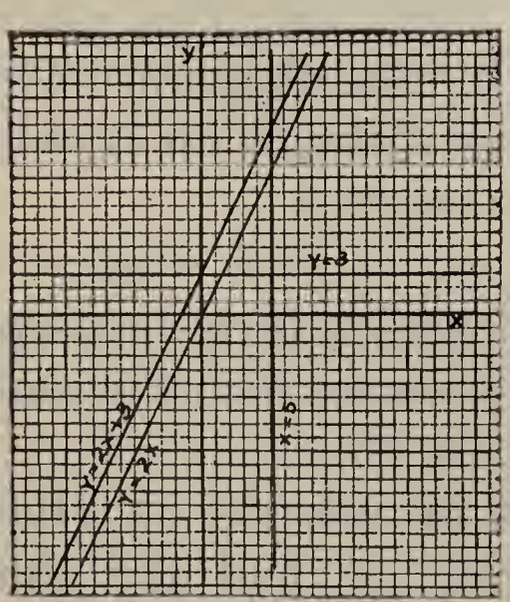

Fig. 133.

equation of the $y-$axis is $x = o$ and $x = 5$ is a straight line five units to the right of the y axis.

All of these equations are of the first degree, that is, the exponents of the unknowns x and y are one in all cases. They are all the forms we can get of the equation $ax + by + c = o$, by making the coefficients a, b, c, positive or negative quantities or zero. So it is evident that every equation of the first degree can be represented by a straight line and conversely every straight line corresponds to an equation of the first degree. From this point on right thru mathematics, geometry and algebra go hand in hand, sometimes one leading, sometimes the other.

3. Principle and Use of Graphs.—As two points are all that is required to fix the position of a straight line, each of these lines could have been plotted by selecting two points and drawing the straight line thru them. The units on the two axes, as the scales are called, are generally the same; but sometimes it is convenient to have them different. This will not change the type of the graph but only shorten or lengthen it proportionally. We can call the two axes x and y if our equation is in x and y. If the equation is in s and t, then one of the axes can be s and the other t and the plotting be done in exactly the same manner.

Such graphs as these are used to represent the change in temperature during the day, the fluctuations in business, the change in population of a city or country, the rise and fall of the temperature of an invalid, the routing of trains on the railroad, the path of a projectile, the amount of rainfall during the year, the increase in products of the same or different countries, the profit per workman in a factory and any other relation that can be expressed in terms of time or any

other single variable. Even when such expressions cannot be made, we plot a number of points by actual observation and draw a smooth curve thru them, giving us a fairly accurate picture of their variations.

4. Systems of Equations.—The equation $3x + 4y = 7$ has an unlimited number of solutions, and it can be shown that the equation $6x + 8y = 14$ has this same set of values. Such equations are said to be *dependent*, as one is obtained from the other by multiplying every term by 2. When all coefficients have the same ratio the equations are dependent. The equation $3x + 4y = 8$ has no values in common with $3x + 4y = 7$, but if we plot these two equations we find that their graphs are the same distance apart everywhere as in the figure. Such lines are *parallel* and therefore never meet, and such equations are called *inconsistent equations*. The coefficients of the x's and y's are in proportion as in *dependent equations*, but the numbers or *constant terms* as they are called, are in a different ratio.

The equation $x + y = 2$ will have only the value $x = 1, y = 1$ in common with $3x + 4y = 7$, as may be seen by plotting Fig. 134. But we will calculate this in another way, as follows:

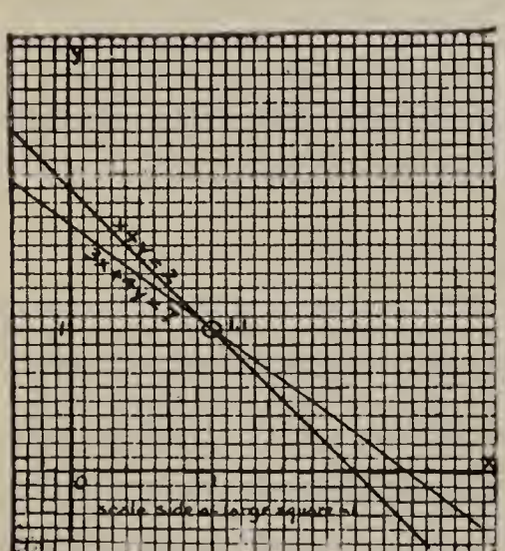

Fig. 134.

$$\left\{\begin{aligned} 3x + 4y &= 7 \\ x + y &= 2 \end{aligned}\right.$$

$$\begin{aligned} 3x + 4y &= 7 \\ 3x + 3y &= 6 \quad \text{(Same as } x + y = 2\text{)} \\ y &= 1 \quad \text{(By subtraction)} \\ x + 1 &= 2 \quad \text{(Substituting 1 for } y\text{)} \end{aligned}$$

$$\text{Therefore } \left\{\begin{aligned} x &= 1 \\ y &= 1 \end{aligned}\right.$$

Note:—*Always put each x with its y.*

5. Solving by Algebra.—It will be instructive to the reader to solve the following equations by both the algebraic and the graphical methods.

$$\begin{cases} x + y = 17 \\ x - y = 7 \end{cases} \quad \begin{cases} x + 5y = 14 \\ 3x - 4y = 4 \end{cases} \quad \begin{cases} 33x + 54y = -9 \\ 44x - 81y = -294 \end{cases}$$

In these equations the coefficients of either x or y should be made alike and it is generally best to use the L. C. M. of the coefficients. In the last example we might multiply the first equation thru by 4 and the second by 3, getting $132x$ in each case. We can *eliminate* either by addition or subtraction as the case may be and in the example we solved $\begin{cases} 3x + 4y = 7 \\ x + y = 2 \end{cases}$ we might have said that as $x + y = 2$, then x equals $2 - y$, whence $3x + 4y = 7$ becomes $6 - 3y + 4y = 7$, whence $y = 1$. And since $x + y = 2$, y also equals 1, giving the same result as before. This method is called *elimination by substitution.* We might also have said as before that $x = 2 - y$ and from the other equation, since $3x = 7 - 4y$, then in that equation $x = \frac{7 - 4y}{3}$. Putting these two values of x equal to each other,

$$2 - y = \frac{7 - 4y}{3}$$

Whence $$6 - 3y = 7 - 4y$$

Hence $$\begin{cases} y = 1 \\ x = 1 \end{cases}$$

We have to use our judgment to determine which method is the more suitable for our problem.

Solve the following problem:

$$\begin{cases} 21x - 23y = 2 \\ 7x - 19y = 12 \end{cases}$$

We could find the value of $21x$ in each equation and put these values equal to each other.

Also solve this problem:

$$\begin{cases} \frac{x}{a} + \frac{y}{b} = 1 \\ \frac{x}{2a} - \frac{y}{3b} = 4 \end{cases}$$

Multiplying the second equation thru by 2, we have

$$\frac{x}{a} - \frac{2y}{3b} = 8$$

Whence $1 - \frac{y}{b} = 8 + \frac{2y}{3b}$; by equating the two values of $\frac{x}{a}$.

$$\frac{b - y}{b} = \frac{24b + 2y}{3b}$$

$$3b - 3y = 24b + 2y$$

$$-21b = 5y$$

Evidently

$$\begin{cases} y = \frac{-21b}{5} \\ x = \frac{26a}{5} \end{cases}$$

A class room has 54 desks, some of which are single and some double. The seating capacity of the room is 72. How many desks of each kind are there?

Let x = number of single desks

y = number of double desks

Then $\begin{cases} x + y = 54, \text{ the number of desks} \\ x + 2y = 72, \text{ (The double desks seat two)} \end{cases}$

As

$$y = 18$$

$$x = 36 \quad \text{(Check the result.)}$$

Going with the current a steamer makes 19 miles per hour, while against the current the speed averages 13 miles per hour. Find speed of current and boat. The equations are

$$\begin{cases} x + y = 19 \\ x - y = 13 \end{cases}$$

letting x = speed of steamer and y = speed of current.

Sometimes there are three unknowns, as in the following:

$$\begin{cases} x + y + 2z = 9 \\ 2y + z = 7 \\ 3x + 5y = 13 \end{cases}$$

$$\begin{cases} x + y + 2z = 9 \\ 4y + 2z = 14 \end{cases}$$

$$x - 3y = -5$$

$$\begin{cases} 3x - 9y = -15 \\ 3x + 5y = 13 \end{cases}$$

$$14y = 28$$

$$\begin{cases} y = 2 \\ z = 3 \\ x = 1 \end{cases}$$

The process here involved is that of reducing from three equations in the same three unknowns to two equations in the same two unknowns and then eventually to one equation in one unknown. The last equation is in x and y, so we will get rid of the z in the first two equations in order to obtain a second equation in x and y to compare with it.

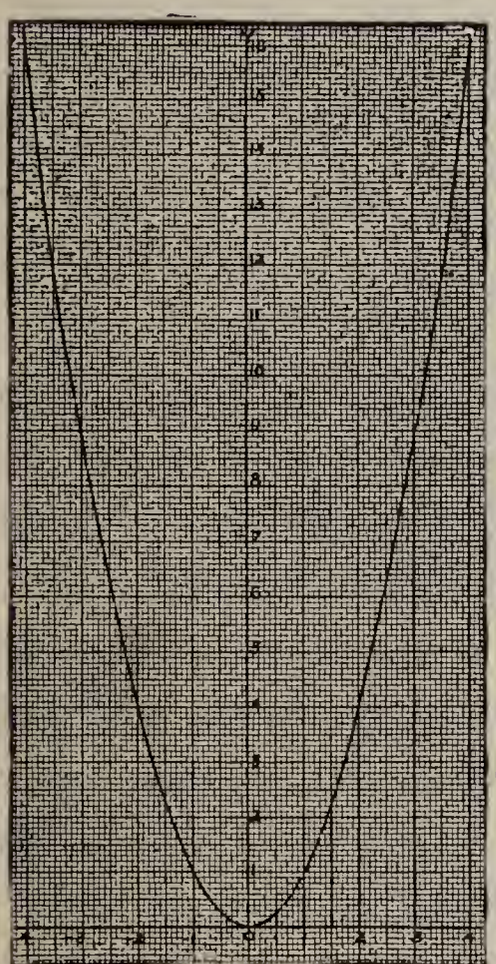

Fig. 135.

6. Parabola $y = x^2$.—The diagram Fig. 135 gives the plot of $y = x^2$, for when $y = 1$, $x = \pm 1$, and when $y = 4$, $x = \pm 2$, etc. Take a piece of squared paper containing large squares whose sides are ten times that of the smaller squares and replot this curve very exactly. (Such paper can be readily purchased at any architectural or engineering supply house.) This curve will give us the values of the squares of all the numbers, whole or decimal, we can represent on the x axis by reading the proper y in each case. Also, we can read the square root of all the numbers represented by the y axis by finding the proper x, in just the same manner as we used our multiplication table. It will be well to call the side of the large square 1, then the side of the smaller square is .1, or one-tenth.

7. Plots of Radical Quantities.—We learned in Ge-

ometry from the great Pythagorean theorem that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides. By this theorem, if the diagonal of a unit square be drawn, it will be in length the $\sqrt{2}$, since $1^2 + 1^2 = 2$ where 2 is the square of the long side. The length of two such diagonals is evidently $2\sqrt{2} = \sqrt{8}$. An accurate pair of dividers set with their points two units apart will enable us to get the $\sqrt{3}$ by placing one point of the dividers on the y axis, one unit above the x axis and describing an arc. This arc will cut the x axis $\sqrt{3}$ distance from the y axis. We can also accomplish this by constructing a right triangle, one of whose legs is 1 and the other $\sqrt{2}$.

Also, $\sqrt{5}$ is the hypotenuse of a right triangle whose legs are 2 and 1; $\sqrt{6}$ can be obtained from a right triangle whose legs are either 1 and the $\sqrt{5}$, 2 and the $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{3}$, or from a right triangle whose hypotenuse is 3 and one leg of which is the $\sqrt{3}$.

The $\sqrt{10}$ is the hypotenuse of a triangle whose sides are 3 and 1. Numerous other square roots will test one's ingenuity. They can be checked on the curve $y = x^2$ (Fig. 135) and will show conclusively that the radical is just as real a number as any other number we may use. We can get $2 + \sqrt{5}$, Fig. 136, by making a line whose length is the sum of 2 and the $\sqrt{5}$. To get $2 - \sqrt{5}$, we take a line two units in length, then measure from the right end with our dividers in a

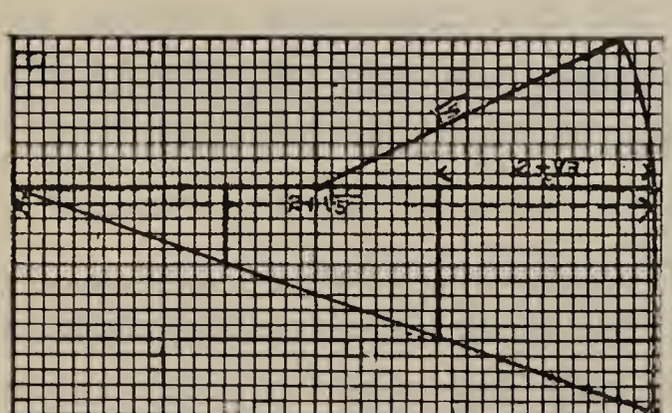

Fig. 136

negative direction the $\sqrt{5}$. The length from the left hand of the first line to the divider point is $2 - \sqrt{5}$. As it is to the left of our beginning point, it is negative. The figure also shows $\frac{2+\sqrt{5}}{3}$.

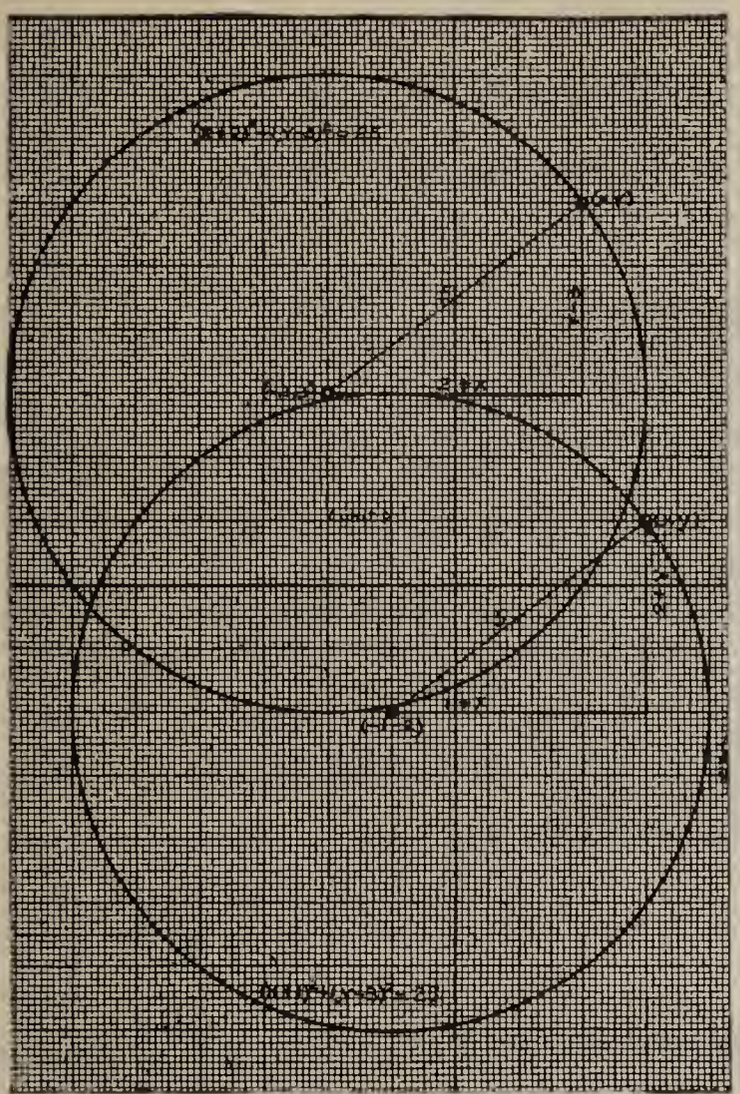

Fig. 137.

8. Graphs of Quadratics.—It could be shown that $x^2 + y^2 = 16$ represents a circle of radius 4 about an origin; and that $x^2 + y^2 = 5$ represents a circle of radius $\sqrt{5}$. In the diagram, Fig. 137, it is shown that $(x + 2)^2 + (y - 3)^2 = 25$ is a circle of radius 5 whose center is (2,3) and that $(x + 1)^2 + (y + 2)^2 = 25$ is the same circle with its center at the point $(-1, -2)$. Whenever we see an equation in two variables of the second degree in which the coefficients of x^2 and y^2 are equal in value and sign we can be sure we have a circle.

We made a careful plot of the parabola $y = x^2$, Fig. 135. Other parabolas are $y^2 = x + y$ and $x = y^2 + 2y - 3$. If no xy term occurs, we distinguish them by the fact that either x^2 or y^2, but not both, is present in the second degree.

We will plot the graph of the equation $x = y^2 + 2y$

-3 (Fig. 138). The following sets of values satisfy the equation:

$$x = -3, 0, -4, 5, -3, 0, \quad 5$$
$$y = \quad 0, 1, -1, 2, -2, 3, -4.$$

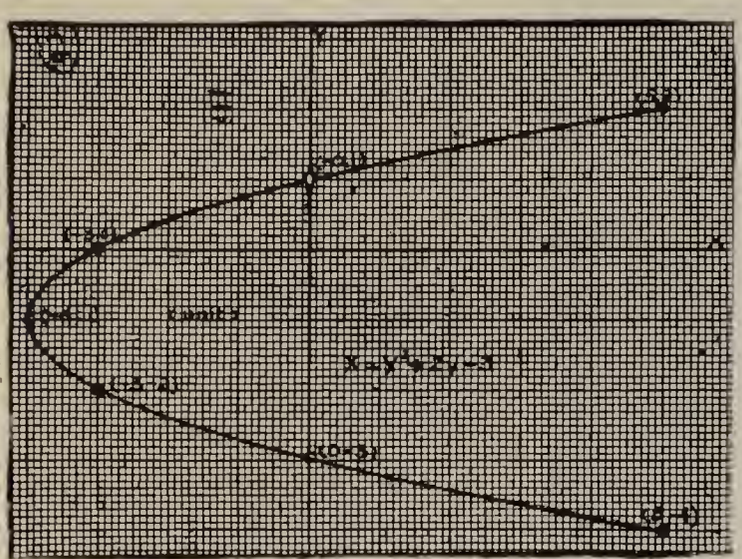

Fig. 138.

Plotting these points and drawing a smooth curve thru them we have the graph of $x = y^2 + 2y - 3$.

9. The Ellipse.—In the equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$, simultaneous values are

$$\begin{cases} x = 0, 5 \\ y = 4, 0 \end{cases}$$

We will plot these values as in Fig. 139. Evidently x cannot be greater than 5 nor y greater than 4, as the other *coordinate*, as it is called, would be imaginary.

Further, if $-x$ is substituted for x it will not change the equation. Hence we say that the curve is symmetrical with respect to the y axis. It will also be found symmetrical with respect to the x axis. Thus we need only to obtain one-fourth of the curve.

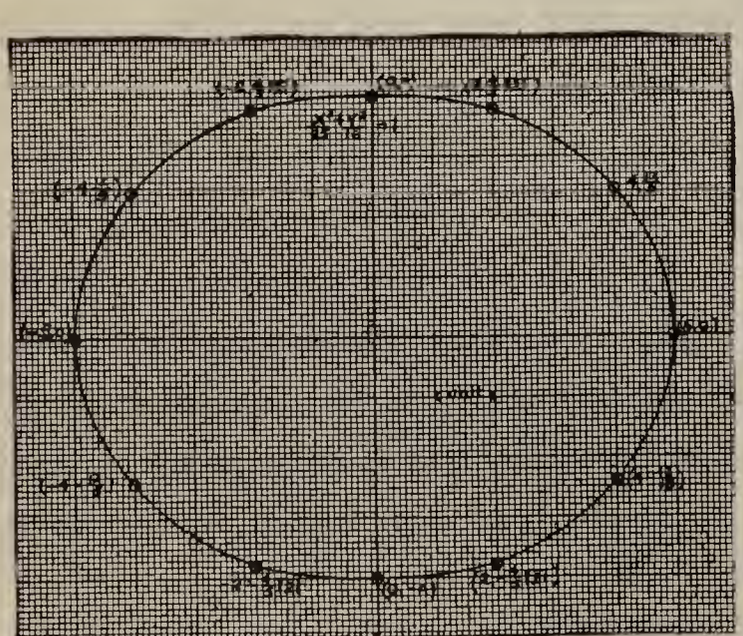

Fig. 139.

Clearing the equation of fractions,

$$16x^2 + 25y^2 = 25 \times 16. \quad \text{Whence}$$

$$25y^2 = 25 \times 16 - 16x^2$$

$$= 16\,(25 - x^2)$$

$$y^2 = \tfrac{16}{25}\,(25 - x^2)$$

$$y = \pm \tfrac{4}{5}\sqrt{25 - x^2}$$

$$\text{If} \quad x = 2,\ y = \pm \tfrac{4}{5}\sqrt{21}.$$

The $\sqrt{21}$ is most easily obtained from a right triangle whose hypotenuse is 5 and one of whose legs is 2. We take $\frac{4}{5}$ of this length, set our dividers and use it as the y of a point whose x is 2.

$$\text{If } x = 4,\ y = \tfrac{4}{5}\sqrt{9} = \pm \tfrac{12}{5}$$

We use $\frac{12}{5}$ as the y of a point whose x is 4. Taking our dividers we set out new y's below the x axis the same length as above, for there is a $-y$ to every $+y$. Likewise we set out new x's to the left of the y axis as there is a $-x$ to every $+x$. A smooth curve thru these points gives us the *ellipse.*

The equation $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{16} = 1$ is the same ellipse with its center at the point (2,3) instead of the origin. It is evidently symmetrical with respect to a line drawn thru (2,3) and parallel to the axis. *Whenever in a second degree equation there is no term in xy, if x^2 and y^2 are present with the same sign and different coefficients we have an ellipse.*

10. The Hyperbola.—In the graph of the equation $\frac{x^2}{25} - \frac{y^2}{16} = 1$, simultaneous values are

$$\begin{cases} x = \pm 5, \pm 6\frac{1}{4}, & \pm 7, & \pm 13 \\ y = \quad 0, \pm 3, & \pm\frac{8}{5}\sqrt{6}, & \pm 9\frac{3}{5} \end{cases}$$

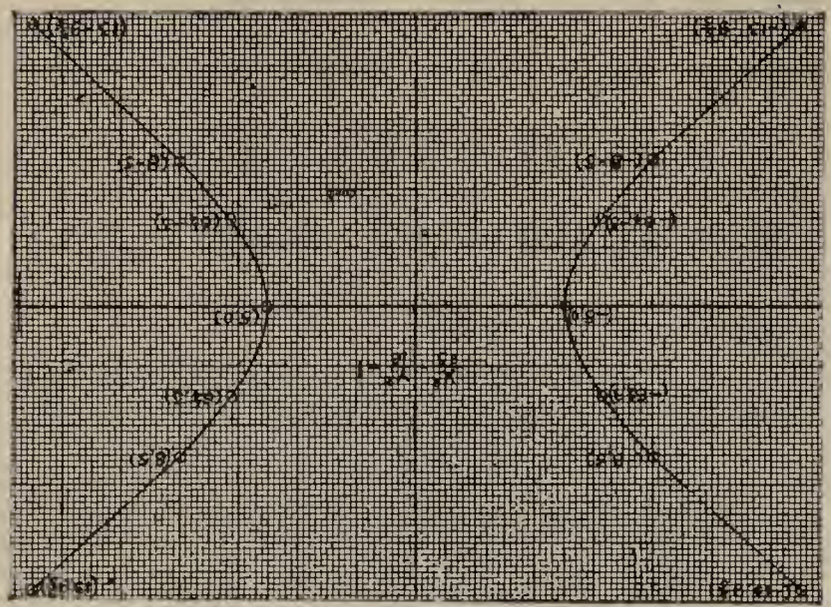

Fig. 140.

Evidently when x is less than 5 numerically y^2 is negative and y is imaginary; also as x increases y increases. The plot is given in the diagram, Fig. 140. *Whenever in a second degree equation, not containing an xy term, x^2 and y^2 are both present with different signs we have an hyperbola.* Fig. 141 shows an hyperbola referred to different sets of axes.

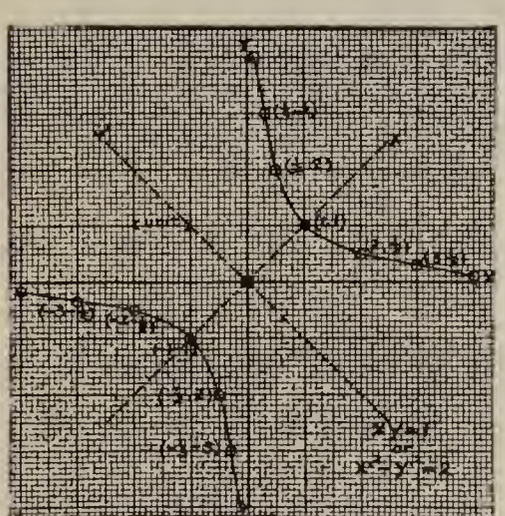

Fig. 141.

The circle parabola, ellipse and hyperbola can be proved to be the only graphs given by second degree equations, except when the equation is factorable, when we have two straight lines. Familiarity with the forms of these curves will aid in plotting the graphs of quadratic equations.

11. Simultaneous Equations.—Let us graph on the same axes the circle $x^2 + y^2 = 25$ and the lines $3x + 4y = 20$, $3x + 4y = 25$ and $3x + 4y = 30$, as shown in Fig. 142. The first line *cuts* the circumference in two distinct points, the second seems to be *tangent* to it and the third *does not meet* it.

In considering the relative positions of such straight lines and curves, we take the literal simultaneous equations

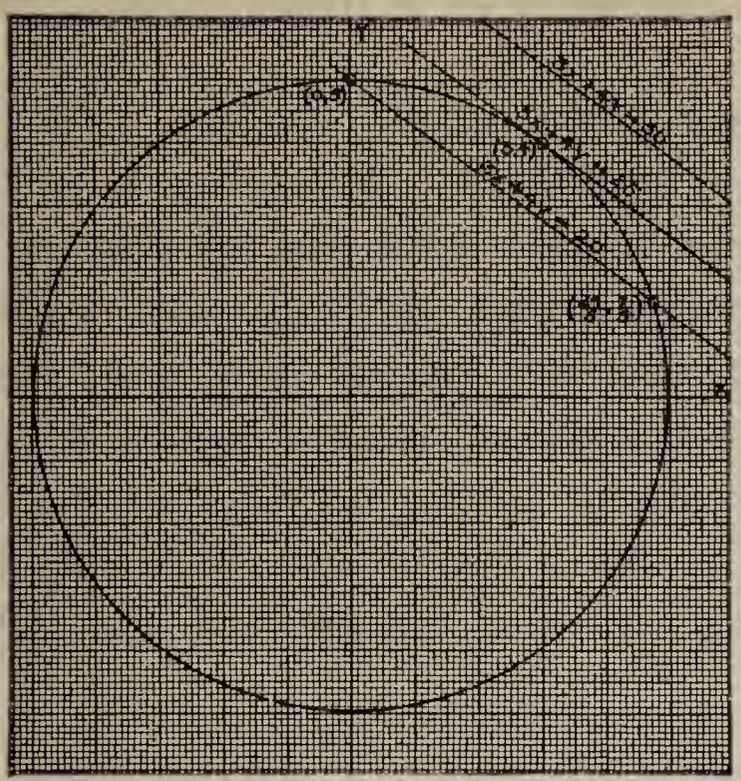

Fig. 142.

$$\begin{cases} x^2 + y^2 = r^2 \\ 3x + 4y = k \end{cases}$$

From the second equation

$$y = \frac{k - 3x}{4}.$$

Hence the first equation becomes

$$x^2 + \left\{\frac{k - 3x}{4}\right\}^2 = r^2$$

Simplifying we get

$$x = \frac{3k \pm 4\sqrt{25r^2 - k^2}}{25}.$$

The two values of x are the *abscissas* or x's of the points of intersection of the circle and the line. As we learned in Algebra, Chapter VII, these values are *real* and *unequal* if the quantity under the radical is positive, real and equal if it is zero and *imaginary* if it is negative. In our graph example (Fig. 142) $r = 5$. Evidently, when $k = 20$ the roots are real and distinct; when $k = 25$ the roots are real and coincident; when $k = 30$ the roots are imaginary. The line $3x + 4y = 20$ cuts the circle where $x = \frac{24}{5}$ or $x = 0$ and by substitution the y's are $\frac{7}{5}$ and 5. The two points of intersection being in this case $\left(\frac{24}{5}, \frac{7}{5}\right)$ and (0,5). In the case of $3x + 4y = 25$, the points come out (3,4) for both values of x and as the line touches the circle in two coincident points it is called tangent. In the last case, $3x + 4y = 30$, the value of x being imaginary shows that the line does not cut the circle at all. As k could be

negative there will be similar sets of lines on the other side of the circle.

This method illustrates the most common type of simultaneous quadratic equations. When we have an equation of the second degree and one of the first, the equations being independent, we begin by finding the value of x or y in the first degree equation, then substitute this value in the second degree equation and solve the resulting quadratic, giving us the two values of one coordinate. The other coordinate can easily be found by substituting in the simpler of the two equations.

12. Examples in Simultaneous Equations.—The great importance of simultaneous quadratic equations calls for one or two examples of their solution.

The two equations $\begin{cases} x^2 - y^2 = 4 \\ x + y = 2 \end{cases}$ appear to be simultaneous quadratics, but they are not independent, as may be seen by dividing equals by equals, when we get $\frac{x^2 - y^2}{x + y} = 2$, whence $x - y = 2$. This latter equation can be taken with either of the others. Of course, we take it with the easier one and get the system $\begin{cases} x - y = 2 \\ x + y = 2 \end{cases}$ evidently giving $x = 2$, $y = 0$. *Always test every system of quadratics for independence.*

Another type of simultaneous quadratics is $\begin{cases} 2x^2 - 3xy + 4y^2 = 3 \\ 3x^2 - 4xy + 3y^2 = 2, \end{cases}$ where all the terms but one are in the second degree in the unknown. Getting the numbers alike, we have $\begin{cases} 4x^2 - 6xy + 8y^2 = 6 \\ 9x^2 - 12xy + 9y^2 = 6, \end{cases}$ subtracting the upper from the lower, we get $5x^2 - 6xy + y^2 = 0$, whence, factoring

$$5x - y = 0 \text{ or } x - y = 0$$

Hence we make *two systems*, taking each of these with the simpler of the original equations.

$$\begin{cases} y = 5x \\ 2x^2 - 3xy + y^2 = 3 \end{cases}$$

$$2x^2 - 15x^2 + 25x^2 = 3$$

$$12x^2 = 3$$

$$x^2 = \frac{1}{12}$$

$$\begin{cases} x = \frac{\pm 1}{2\sqrt{3}} \\ y = \frac{\pm 5}{2\sqrt{3}} \end{cases}$$

$$\begin{cases} y = x \\ 2x^2 - 3xy + y^2 = 3 \end{cases}$$

$$2x^2 - 3x^2 + 4x^2 = 3$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$\begin{cases} x = \pm 1 \\ y = \pm 1 \end{cases}$$

There will always be as many answers as the product of the degrees of the two equations, providing the systems are independent.

Only special forms of simultaneous quadratics admit of comparatively easy solutions.

13. Special Devices.—Tho most quadratic simultaneous equations may be solved by the two methods in the preceding section, there are a few special devices of considerable value.

The system $\begin{cases} (2x - 3y + 1)(3x - y - 2) = 0 \\ (x - 2y + 8)(5x + 6y + 1) = 0 \end{cases}$

offers no difficulty, for evidently either of the first two factors can be associated with either of the second two factors, thus making four sets of first degree systems, as follows:

$$\begin{cases} 2x - 3y + 1 = 0 \\ x - 2y + 8 = 0 \end{cases} \quad \text{or} \quad \begin{cases} 2x - 3y + 1 = 0 \\ 5x + 6y + 1 = 0 \end{cases}$$

or

$$\begin{cases} 3x - y - 2 = 0 \\ x - 2y + 8 = 0 \end{cases} \quad \text{or} \quad \begin{cases} 3x - y - 2 = 0 \\ 5x + 6y + 1 = 0 \end{cases}$$

The system $\begin{cases} \frac{1}{x} + \frac{1}{y} = 5 \\ \frac{1}{x^2} + \frac{1}{y^2} = 13 \end{cases}$ becomes readily soluble when we put

$$\frac{1}{x} = a \text{ and } \frac{1}{y} = b.$$

The system $\begin{cases} x^2 + y^2 + x + y = 8 \\ xy - 2 \end{cases}$ can be expressed in better form by adding twice the second equation to the first, this giving

$x^2 + 2xy + y^2 + x + y = 12$. Then let

$x + y = b$, which reduces the equation to

$$b^2 + b = 12$$

Whence $b = 3$ or -4,

Hence $x + y = 3$ or $x + y = -4$ each of which can be taken with $xy = 2$.

The system $\begin{cases} x^4y^4 + x^2y^2 = 272 \\ x^2 + y^2 = 10 \end{cases}$ can be solved by noticing that x^4y^4 is the square of x^2y^2 and that the first equation is in the form of a quadratic, which being solved gives $x^2y^2 = 16$ or -17. Then xy can be found and hence x and y by substitution in $x^2 + y^2 = 10$.

14. Geometry With Algebra.—We will parallel the algebraic and graphical solution of a simple pair of simultaneous quadratics, relating each equation in turn to its graph. Let a system be

$$\begin{cases} x^2 + y^2 = 25 \\ xy = 12 \end{cases}$$

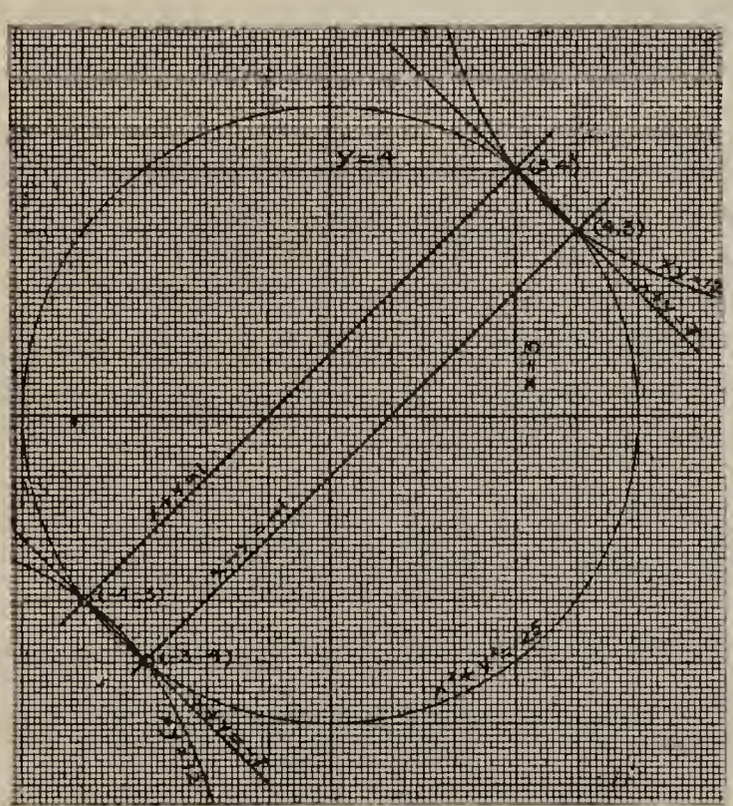

Fig. 143.

A simple algebraic solution is obtained by adding twice the second equation to the first, getting

$$x^2 + 2xy + y^2 = 49$$

Whence $x + y = \pm 7$

Likewise, subtracting twice the second equation

from the first and simplifying gives $x - y = \pm 1$. Plotting these straight lines as in Fig. 143, we see that they intersect in the same places that the original curves intersect. It will take all four systems possible to give the four sets of answers, but no two equations that are factors of the same equation should be taken together. The four solutions are

$$x = 3, -3, 4, -4,$$

$$y = 4, -4, 3, -3.$$

Plotting these lines, we see that the lines $x = 3$ crosses the lines $y = 4$ at the place where the original curve crossed, etc., and it is now evident that these different systems of equations are geometrically shown equivalent to the original equations.

A careful study of graphs of simultaneous quadratics in conjunction with their algebraic solution is necessary to any adequate understanding of the latter and, tho long and tedious at times, is well worth many hours of careful study by any one who desires a foundation on which to build.

15. Synthetic Division.—The following problem is already familiar to us and is reproduced here in order that the reader may more readily understand what follows:

$$\begin{array}{r|l} 2x^3 - 5x^2 + 3x - 4 & x - 3 \\ \underline{2x^3 - 6x^2} & \\ x^2 + 3x & 2x^2 + x + 6 \\ \underline{x^2 - 3x} & \\ 6x - 4 & \\ \underline{6x - 18} & \\ 14 & \end{array} \qquad \begin{array}{r|l} 2 - 5 + 3 - 4 & 1 - 3 \\ \underline{2 - 6} & \\ 1 + 3 & 2 + 1 + 6 \\ \underline{1 - 3} & \\ 6 - 4 & \\ \underline{6 - 18} & \\ 14 & \end{array}$$

This problem can be worked in a much shorter way. The first term of the divisor might easily be omitted

without causing confusion and it is very evident that we do not need anything but the first terms of each partial remainder. The partial products have their first term the same as those of the partial remainders, which have been crossed out. It will be noticed that it is unnecessary to write the quotient, for the coefficients of the quotient are the first terms of the successive remainders. Also, since subtraction can be performed by changing the signs and adding we will change -3 into $+3$. The work will then be as follows:

$$\begin{array}{r} 2-5+3-4 \mid +3 \\ \underline{6} \\ 1 \\ \underline{+3} \\ 6 \\ \underline{+18} \\ 14 \end{array}$$

In this we say, in practice, $3 \times 2 = 6$, which added to -5 gives 1; $3 \times 1 = 3$, which added to 3 gives 6; $3 \times 6 = 18$, which added to -4 gives 14.

Arranging this to take up less space we have

$$\begin{array}{r} 2-5+3-4 \mid 3 \\ \underline{+6+3+18} \\ 2+1+6+ \mid 14 \end{array}$$

The answer of $2x^3 - 5x^2 + 3x - 4$ divided by $x - 3$ is $2x^2 + x + 6$ with a remainder of 14. From the factor theorem, in algebra, if $f(x) = 2x^3 - 5x^2 + 3x - 4$ then $f(3) = 14$. This method is called *synthetic division.*

16. Theory of Equations.—The man who said that as a boy he daily swam three times across the creek near his home was embarrassed when asked how he got back after the third crossing. It is just as evident that if $f(a)$ and $f(b)$ have opposite signs the function

must have been 0, an odd number of times between $x = a$ and $x = b$. We may use this principle to locate roots and synthetic division will enable us appreciably to shorten our process.

Let us obtain graphically the roots of $f(x) = 2x^3 - 5x^2 + 3x - 4 = 0$. First we will make a table of simultaneous values of x and y, where $y = f(x)$. Here is the table:

x	y
3	14
2	− 2*
1	− 4
0	− 4
− 1	−14
$\frac{1}{2}$	$-3\frac{1}{2}$

Then we proceed as follows:

$$\begin{array}{r|l} 2-5+3-4 & 1 \\ \quad 2-3+0 & \\ \hline 2-3+0 & -4 \end{array}$$

$$\begin{array}{r|l} 2-5+3-4 & 2 \\ \quad 4-2+2 & \\ \hline 2-1+1 & -2 \end{array}$$

$$\begin{array}{r|l} 2-5+3-4 & 3 \\ \quad +6+3+18 & \\ \hline 2+1+6 & +14 \end{array}$$

$$\begin{array}{r|l} 2-5+3-4 & -1 \\ \quad -2+7-10 & \\ \hline 2-7+10 & -14 \end{array}$$

$$\begin{array}{r|l} 2-5+3-4 & \frac{1}{2} \\ \quad +1-2+\frac{1}{2} & \\ \hline 2-4+1 & -3\frac{1}{2} \end{array}$$

$$\begin{array}{r|l} 2-5+3-4 & 2.1 \\ \quad 4.2-1.7+2.8 & \\ \hline 2-.8+1.3-1.2 & \end{array}$$

*Notice the change in sign, showing that the root is between 2 and 3. The root is 2 and some decimal. There are no larger roots than 3. The terms being all + they will become larger for successive values of x and there will, therefore, be no further change in sign.

Now that the signs alternate, there will be no smaller negative root, for the coefficients merely increase. As there are two successive values of $y = -4$ in the plot we must get y for $x = \frac{1}{2}$, a value in between 0 in 1.

$$
\begin{array}{r|l}
2 - 5 + 3 - 4 & 2.2 \\
\quad 4.4 - 1.3 + 3.7 & \\
\hline
2 - .6 + 1.7 & -.3
\end{array}
$$

$$
\begin{array}{r|l}
2 - 5 + 3 - 4 & 2.3 \\
\quad 4.6 - .9 + 4.8 & \\
\hline
2 - .4 + 2.1 + & .8
\end{array}
$$

and the curve appears as in the graph, Fig. 144.

Fig. 144.

The only real root is between 2 and 3 and evidently it is nearer 2 than 3. We will plot the part of the curve between $x = 2$ and $x = 3$ on a larger scale and get the next figure of the root. This magnifying can be repeated as often as desired.

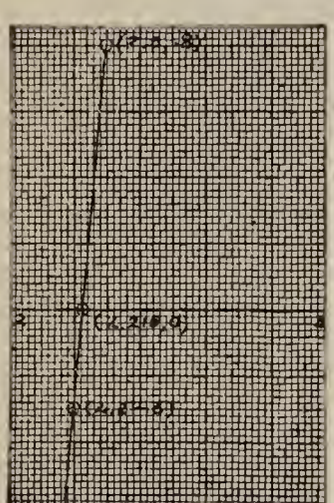

Fig. 145.

As there is a change in sign in (x) the root is between 2.2 and 2.3. If the plot is magnified as in accompanying graph, Fig. 145 by taking units 10 times as large, a very good estimate of the next figure can be made by drawing a straight line and thus saving much computation. The root is 2.215 to three places.

Sometimes more than one root will appear between two successive integers. The plot, Fig. 144, shows this, the "kink" in the curve indicating the presence of imaginary roots when the x axis is not crossed.

As another example of synthetic division we will write $2x^3 - 5x^2 + 3x - 14$ in powers of $(x - 2)$.

$$
\begin{array}{r|l}
2 - 5 + 3 - 4 & 2 \\
\quad + 4 - 2 + 2 & \\
\hline
2 - 1 + 1 & +2 \\
\quad + 4 + 6 & \\
\hline
2 + 3 & +7 \\
\quad + 4 & \\
\hline
2 + & 7
\end{array}
$$

We are dividing by $x - 2$, which goes $2x^2 - x + 1$ times. That is, there are $2x^2 - x + 1$ of the $(x - 2)$'s and a remainder of -2. Dividing $2x^2 - x + 1$ by $(x - 2)$ we get $2x + 3$, the number of $(x - 2)^2$, and the remainder is 7 of the $(x - 2)$'s. Dividing $2x + 3$ by $(x - 2)$ we get 2, the number of the $(x - 2)^3$ and a remainder of $7(x - 2)^2$'s. We can say

$2x^3 - 5x^2 + 3x - 4 = 2(x-2)^3 + 7(x-2)^2 + 7(x-2) - 2$. Expand this as a check. The roots of $2y^3 + 7y^2 + 7y - 2 = 0$ are two less than those of $2x^3 - 5x^2 + 3x - 4 = 0$. Here $y = (x-2)$.

17. Relations of Roots and Coefficients.—If an equation has three roots a, b and c, then the equation has the factors $x-a$, $x-b$, and $x-c$, each equal to 0. The equation is evidently $(x-a)(x-b)(x-c) = 0$. Then $(x-a)(x-b)$

$=$ $x^2 - (a+b)x + ab$ and multiplying this
by $(x-c)$
we have
$$x^3 - (a+b)x^2 + abx$$
$$-\quad c \quad x^2 + (ac+bc)x = abc$$
$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = 0$$

We notice that as in the quadratics the coefficient of the second term is $-$ (the sum of the roots) and the coefficient of the second term is $+$ (the sum of the roots) taken two at a time. The coefficient of the fourth term has them three at a time. This law is generally true and could have been easily proved by means of combinations. It will serve as an excellent check.

A reversal of this process would show us that if an equation has one root it must have as many as its degree. The fundamental theorem of algebra that every equation of the form $f(x) = 0$ has at least one root is too long and difficult to be given here.

REVIEW.

1. What is meant by plotting a graph?
2. What are some uses of graphs?
3. How would you recognize a parabola?
4. When are two equations simultaneous? When are they independent?
5. Give one of the special devices for solving simultaneous quadratic equations.

CHAPTER XIV

DIFFERENTIAL CALCULUS

1. Fundamental Principles and Formulas.—Suppose we had two variables so related that a change in one produces a change in the other. It would be very desirable to know the relationship between these changes. For instance, does a rise in the selling price produce more income? To examine such a problem in all its bearings necessitates our establishing the relationship between the change of the function and the change of the independent variable, so that we know how they compare, as we take infinitesimally small changes in the two variables. The limit of this ratio is called the *derivative.* A speedometer on an automobile indicates derivatives. This expression also means the rate of change of the two variables at any instant, such as is illustrated by the fact that tho a suburban train never travels 40 miles in any one hour, it may often develop a speed of more than 40 miles an hour.

We will consider the function $y = x^2$, already familiar to us, and assume an initial value of x, say 4. The initial value of y is evidently 16. If the increment of x (indicated by the symbol Δ, meaning increment or change, and written Δx), be 1 then the new value of x is 5. Hence the new value of y (since y is x^2) is 25 and as the initial value of y was 16, the increment of y, (Δy) is 9 and $\frac{\Delta y}{\Delta x} = \frac{9}{1} = 9$.

2. Ratio of Increments.—The following table should be computed, then studied and thought over repeatedly.

It is of great importance, for it shows the behavior of the ratio of the increment of y to the increment of x as the increment of x diminishes.

Table of Ratio of Increments.

Initial value of x	New value of x	Increment Δx	Initial value of y	New value of y	Increment Δy	$\frac{\Delta y}{\Delta x}$
4	5.0	1.0	16	25.	9.	9.
4	4.8	0.8	16	23.04	7.04	8.8
4	4.6	0.6	16	21.16	5.16	8.6
4	4.4	0.4	16	19.36	3.36	8.4
4	4.2	0.2	16	17.64	1.64	8.2
4	4.1	0.1	16	16.81	0.81	8.1
4	4.01	0.01	16	16.0801	0.0801	8.01

Note $\frac{\Delta y}{\Delta x}$ seems to approach 8 as a limit.

The limit $\frac{\Delta y}{\Delta x}$ could have been obtained more generally as follows: Since $y = x^2$, if x has any initial value, x_1, we will obtain

$$y_1 = {x_1}^2$$

Now let x take on a change Δx, when the new x will be $x_1 + \Delta x$, and since this will probably make a change in y we have a new y equal to $(y_1 + \Delta y)$, where Δy can be obtained from the equation

$$y_1 + \Delta y = (x_1 + \Delta x)^2$$

Expanding $$y_1 + \Delta y = {x_1}^2 + 2x_1\Delta x + (\Delta x)^2$$

but $$y_1 = {x_1}^2$$

hence $$\Delta y = 2x_1\Delta x + (\Delta x)^2$$

whence $$\frac{\Delta y}{\Delta x} = 2x_1 + \Delta x$$

Since these two quantities are always equal they have the same limit and

$$\lim_{\Delta x = 0} \left[\frac{\Delta y}{\Delta x}\right] = 2x_1.$$

Expressed briefly this is $\frac{dy}{dx} = 2x$.

The reader should commit to memory the fact that $\frac{dy}{dx}$ is only an abbreviation for the expression written above it. It is not a fraction, but the limit of a ratio. We should always think and say the full expression and, further, should remember that the x_1 is important as it is always the derivative at a point.

If the initial value of x be 4, as in the table, the value of $\frac{dy}{dx}$ is 8. This seems to be the value that $\frac{\Delta y}{\Delta x}$ is approaching in the table. Had the initial value of x been 5, $\frac{dy}{dx}$ would be 10.

All derivatives could be worked out by the following four-step rule: *First, replace* x_1 by $x_1 + \Delta x$ giving a new value of y, $(y_1 + \Delta y)$; *second, subtract* the initial value from the new value to find Δy; *third, divide* Δy by Δx and obtain an expression for $\frac{\Delta y}{\Delta x}$, and *fourth, find the limit* of this ratio when Δx varies and approaches the limit 0. This limit is the derivative when $x = x_1$ *any particular* value of x.

3. Fundamental Formulas.—The following is a set of formulas derived in turn by this four-step method and used instead of it for computing derivatives. Here c, n, a and e represent constants, e being the base of the natural system of logarithms; x is the independent variable and u, v, w are functions of x.

(1) $\frac{dc}{dx} = 0$ Thus, $\frac{d}{dx}(116c) = 0$, since there is no change in a constant.

(2) $\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$

(3) $\frac{d}{dx} uv = u\frac{dv}{dx} + v\frac{du}{dx}$ (3′) $\frac{d}{dx} cv = c\frac{dv}{dx}$

(4) $\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (4′) $\frac{d}{dx}\left(\frac{c}{v}\right) = \frac{-c}{v^2}\frac{dv}{dx}$

(5) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (5′) $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

(6) $\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}$ (6′) $\frac{d}{dx} \sqrt{u} = \frac{\frac{du}{dx}}{2\sqrt{u}}$

(7) $\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$

(2) The derivative of a sum is the sum of the derivatives.

(3) The derivative of a variable product of two variables is the first times the derivative of the second times, etc.

(4) The derivative of a fraction is the denominator into the derivative of the numerator minus the numerator into the derivative of the denominator, over the denominator squared.

(6) The derivative of the square root is the derivative of the thing over twice the square root of the thing.

(8) $\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$

(9) $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$

(10) $\frac{d}{dx} \arcsin u = \frac{\frac{dx}{dx}}{\sqrt{1-u^2}}$

(11) $\frac{d}{dx} \arctan u = \frac{\frac{du}{dx}}{1+u^2}$

(12) $\frac{d}{dx} (\log_a u) = \log_a e \frac{\frac{du}{dx}}{u}$ (12′) $\frac{d}{dx} \log_e u = \frac{\frac{du}{dx}}{u}$

(13) $\frac{d}{dx} (a^u) = a^u \log_e a \frac{du}{dx}$ (13′) $\frac{d}{dx} e^u = e^u \frac{du}{dx}$

(14) $\frac{d}{dx} (u^v) = vu^{v-1} \frac{du}{dx} + \log_e u \cdot u^v \frac{dv}{dx}$

Notice that in all these derivatives we *always* end by taking the derivative of the variable in respect to the independent variable. Beginners often forget the $\frac{du}{dx}$ at the end.

4. Examples in Fundamental Formulas.—The following examples show the method of applying the formulas given in the foregoing section:

(1) If $5u^4 + u^3 = 6y$, then, taking derivatives, this is first a sum and formula (2) should be applied. If we are taking derivatives with respect to x, we will get $\frac{d}{dx}(5u^4) + \frac{d}{dx}(u^3) = \frac{d}{dx}(6y)$. Next, formula (3) applies and $5\frac{d}{dx}(u^4) + \frac{d}{dx}(u^3) = 6\frac{d}{dx}(y)$, after which formula (6) applies and $20u^3\frac{du}{dx} + 3u^2\frac{du}{dx} = 6\frac{dy}{dx}$.

(2) If $y = \sqrt{3x^2 - 4x}$, formula (6′) applies and since $\frac{d}{dx}(3x^2 - 4x) = 6x - 4$, we have $\frac{dy}{dx} = \frac{6x - 4}{2\sqrt{3x^2 - 4x}} = \frac{3x - 2}{\sqrt{3x^2 - 4x}}$.

(3) If $y = \frac{5x + 2}{3x - 7}$ then $\frac{dy}{dx} = \frac{(3x - 7)\frac{d}{dx}(5x + 2) - (5x + 2)\frac{d}{dx}(3x - 7)}{(3x - 7)^2}$

by formula (4), hence $\frac{dy}{dx} = \frac{(3x - 7)\,5 - (5x + 2)\,3}{(3x - 7)^2} = \frac{-41}{(3x - 7)^2}$

These problems end with $\frac{dx}{dx}$, which is evidently 1.

(4) Derivative of $\sin(3x^2 = 7) = \cos(3x^2 - 7)\frac{d}{dx}(3x^2 - 7) = -6x\cos(3x^2 - 7)$. Derivative of $\sin^3(3x^2 - 7) = 3\sin^2(3x^2 - 7)\cos(3x^2 - 7)\,6x$, since the derivative of $\sin^3 u = 3\sin^2 u\frac{d}{dx}(\sin u) = 3\sin^2 u\cos u\frac{du}{dx}$. Hence the derivative of $\sin^3(3x^2 - 7) = 18x\sin^2(3x^2 - 7)\cos(3x^2 - 7)$.

(5) $\frac{d}{dx}\arcsin(3x^2 - 7) = \frac{6x}{\sqrt{1 - (3x^2 - 7)^2}}$

(6) $\frac{d}{dx}\log(3x^2 - 7) = \frac{6x}{3x^2 - 7}$.

To differentiate a logarithm of an expression we put the expression in the denominator, and its derivative in the numerator. This type of differentiation abbreviates many otherwise long problems. For instance

(5) if $y = \frac{\sqrt{(3x - 7)(2x + 3)}}{\sqrt[3]{(2x - 6)}}$ we can say from properties of

logarithms that $\log y = \frac{1}{2} \log (3x-7) + \frac{1}{2} \log (2x+3) - \frac{1}{3} \log (2x-6)$, hence

$$\frac{\frac{dy}{dx}}{y} = \frac{3}{2(3x-7)} + \frac{2}{2(2x+3)} - \frac{2}{3(2x-6)} = \frac{48x^2-226x+174}{6(3x-7)(2x+3)(2x-6)},$$

whence $\frac{dy}{dx} = y \frac{48x^2-226x+174}{6(3x-7)(2x+3)(2x-6)}$

$$= \frac{(3x-7)^{\frac{1}{2}}(2x+3)^{\frac{1}{2}}(48x^2-226x+174)}{(2x-6)^{\frac{1}{3}}(3x-7)(2x+3)(2x-6)} = \frac{(48x^2-226x+174)}{6(3x-7)^{\frac{1}{2}}(2x+3)^{\frac{1}{2}}(2x-6)^{\frac{4}{3}}}$$

(8) If $z = 10^{7x^2-6}$ $\frac{dz}{dx} = 10^{7x^2-6} \log_{10} e \frac{d}{dx} (7x^2 - 6) = 14x \cdot 10^{7x^2-6} \log_{10} e$ 10. Usually these problems are easier after taking logarithms of both sides.

(7) This problem can be solved in similar way to the following more difficult problems $y = (x+3)^{x-2}$

Here $\log y = (x-2) \log (x+3)$ and $\frac{\frac{dy}{dx}}{y} = \frac{(x-2)}{(x+3)} + \log (x+3)$

Hence $\frac{dy}{dx} = (x+3)^{x-2} \left[\frac{(x-2)}{(x+3)} + \log (x+3) \right]$

5. Derivation of Formulas.—The reader should establish a foundation for himself by repeatedly working out in full the derivation of these fourteen formulas. We will here derive three of them, each presenting some new combination of former ideas.

$$(7) \quad \frac{d}{du} (\sin u) = \lim_{\Delta u = 0} \left[\frac{\sin (u_1 + \Delta u) - \sin u_1}{\Delta u} \right]$$

$$= \lim_{\Delta u = 0} \left[\frac{\sin u_1 \cos \Delta u + \cos u_1 \sin \Delta u - \sin u_1}{\Delta u} \right]$$

$$= \lim_{\Delta u = 0} \left[\cos u_1 \frac{\sin \Delta u}{\Delta u} - \sin u_1 \left(\frac{1 - \cos \Delta u}{\Delta u} \right) \right]$$

$$= \cos u_1 \text{ Since } \left[\lim_{\Delta u = 0} \frac{\sin \Delta u}{\Delta u} \right] = 1 \text{ and } \cos \Delta u \doteq 1$$

Therefore $\frac{d}{dx} (\sin u) = \cos u \frac{du}{dx}$ by formula (5).

(10) If $y = \arcsin u$

$\sin y = u$

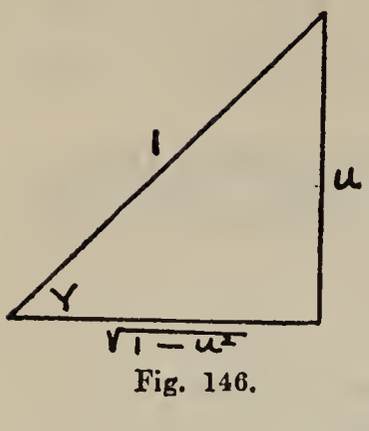

Fig. 146.

Taking derivatives $\cos y \frac{dy}{dx} = \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\cos y} = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}$$ (see Fig. 146).

(12) $y = \log_a u$

$$\Delta y = \log_a (u + \Delta u) - \log_a u = \log_a \left(\frac{u+\Delta u}{u}\right) = \log_a \left(1 + \frac{\Delta u}{u}\right)$$

(Note that when we subtract logarithms we divide numbers.)

$$\frac{\Delta y}{\Delta u} = \frac{1}{\Delta u} \log_a \left(\frac{1+\Delta u}{\Delta u}\right) = \log_a \left(\frac{1+\Delta u}{u}\right)^{\frac{1}{\Delta u}} = \frac{1}{u} \log_a \left(\frac{1+\Delta u}{u}\right)^{\frac{u}{\Delta u}}$$

Note that $k \log x = \log x^k$. The complicated expression $\left(\frac{1+\Delta u}{u}\right)^{\frac{u}{\Delta u}}$ when we put n for $\frac{\Delta u}{u}$, becomes $(1+n)^{\frac{1}{n}}$ and therefore the limit of either is our old friend e. Hence $\frac{d}{du}(\log_a u) = \frac{1}{u} \log_a e$.

6. Other Forms of Notation.—If $y = F(x)$, $\frac{d}{dx}[F(x)]$ is written $F^1(x) = \frac{dy}{dx} = D_x y$.

In obtaining a geometrical significance of the derivative as the slope of the tangent we proceed as follows, referring to Fig. 147. After reading the first line we read vertically and then read the last line, which consists of limits of the expressions in the upper line.

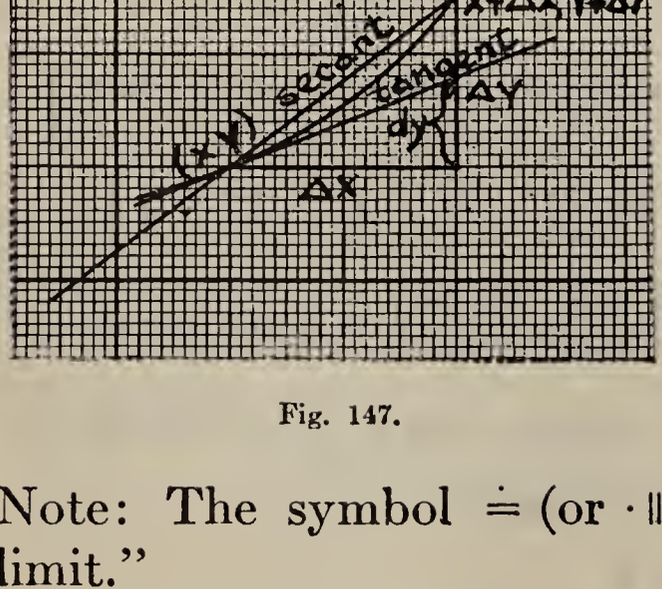

Fig. 147.

$\frac{\Delta y}{\Delta x}$ = slope of secant

$\cdot \| \qquad \cdot \|$

$\frac{dy}{dx}$ = *slope of tangent*

Note: The symbol $\doteq$ (or $\cdot \|$) means "approaches as a limit."

In Section 2 of this chapter we had $\Delta y = 2x_1\Delta x + (\Delta x)^2$, where Δx is the *principal infinitesimal* and $(\Delta x)^2$ is infinitesimal in respect to Δx or, as we say, *of a higher order.* Letting $(\Delta x)^2$ approach 0 as a limit, we have $dy = 2x_1dx$. Note that here dx has a value. With this understanding dy is the principal part of y and is called the differential of y. The diagram, Fig. 147, also shows that the differential is what the increment would be if the slope of the curve became uniform. We can write $dy = F^1(x)\,dx$. *Thus the differential of any function is equal to its derivative multiplied by the differential of the independent variable.* This distinction between the differentials and derivatives is fundamental and the reader should put the fourteen formulas into the differential form so that he may have them for comparison with the formulas of integral calculus.

There is another approach to the calculus called *rates*, where the derivative is taken in respect to the time, t. If velocity, $v = 32t$, acceleration, which is defined as *rate of change of velocity per second per second*, is obtained by taking the derivative of the velocity in respect to the time, thus $\frac{dv}{dt} = 32$.

7. Successive Differentiation.—The expression $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written $\frac{d^2y}{dx^2}$ and is read "the second derivative of y in respect to x."

If $y = x^5$, $\frac{dy}{dx} = 5x^4$, $\frac{d^2y}{dx^2} = 20x^3$, $\frac{d^3y}{dx^3} = 60x^2$.

If $y = \sin x$, $\frac{dy}{dx} = \cos x$, $\frac{d^2y}{dx^2} = -\sin x$, $\frac{d^3y}{dx^3} = -\cos x$, $\frac{d^4y}{dx^4} = \sin x$.

If $y = e^x$, $\frac{dy}{dx} = e^x$, $\frac{d^2y}{dx^2} = e^x$, $\frac{d^3y}{dx^3} = e^x$. Note the *periodicity* of the last two examples.

If $f(x)$ represents the length of an ordinate of any

point on a curve, where the abscissa is x, $f'(x)$ represents the slope of the tangent. Where $f'(x) = 0$ the curve is parallel to the x axis. Where $f'(x) = \pm$ the ordinates are growing either larger or smaller as the case may be, that is, the function is increasing or decreasing. Evidently $f''(x)$ represents the change of the slope of the tangent, that is, the curvature with respect to the x axis.

8. Maxima and Minima.—A continuous curve is horizontal at three different kinds of points. In Fig. 148, A is a maximum, C a minimum, and B and D points of inflection. A motor traveling along a road would find a maximum at every hill crest, a minimum at every valley and possibly many points of inflection on the slopes. At A the ordinates have just stopped increasing and will immediately start to decrease, that is, $\frac{dy}{dx} = +$ before, 0 at, and $-$ after A, while at C the reverse is true. At B and D, $\frac{dy}{dx}$ has the same sign both before and after. (*Note:* Maximum only means greatest in its immediate "neighborhood.")

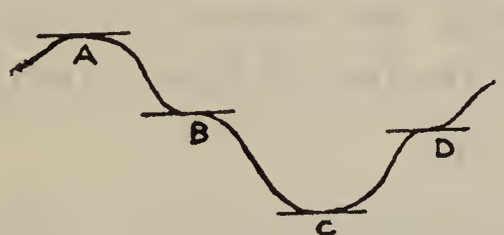

Fig. 148.

We could settle these questions from the second derivative and tell whether we had a maximum or minimum, providing the second derivative were not also 0. For near a maximum point $\frac{dy}{dx}$ is plus, then 0, then minus, hence it decreases and therefore $\frac{d^2y}{dx^2}$ is $-$ at maximum points and $+$ at minimum points. If $\frac{d^2y}{dx^2}$ is 0, we get $\frac{d^3y}{dx^3}$ and if this is $+$ we have a point of inflec-

tion like that at D, if $-$ a point of inflection like that at B. If $\frac{d^3y}{dx^3} = 0$, the theory continues for the next derivative as it did for $\frac{d^2y}{dx^2}$, etc. *When successive derivatives are* 0, *the first* one not 0, if even, gives a minimum when $+$, a maximum when $-$; *if odd, a point of inflection like D when* $+$, *like B when* $-$.

9. Method of Working Problems.—As an example of working a problem in maxima we will take the historical problem "How must a straight line be divided so that the product of its parts is a maximum?"

(1) Suppose 12 be the length of the line, x one part, and $12 - x$ the other. The expression to be examined $f(x) = x(12 - x) = 12x - x^2$

$$f'(x) = 12 - 2x$$
$$f''(x) = -$$

Put $12 - 2x = 0$, $x = 6$, a *critical value;* $f''(6) = -$ (in fact in this problem any $f''(x) = -$). Therefore 6 gives a maximum, which in this case is $6(12 - 6) = 36$. No other partition of 12 will give so large a product. We could have said that as there must be a maximum and as there is only one critical value 6, then the 6 must give the maximum.

(2) What are the most economical dimensions for a quart tomato can? This means a minimum amount of tin, therefore a minimum amount of surface.

Let $S = 2\pi r^2 + 2\pi rh$, where r = radius of base and h height. There are two variables, r and h, but as we only know how to use one we must necessarily eliminate one or the other. As we know that the volume, $V = \pi r^2 h$, then $h = \frac{V}{\pi r^2}$, hence $S = 2\pi r^2 + \frac{2V}{r}$. And as we wish to find the first derivative and put it equal to 0, the constant factor 2 will divide out. So we will drop this constant, as all constant factors are dropped in the beginning, and say

$$f(r) = \pi r^2 + \frac{V}{r}$$

$$f'(r) = 2\pi r - \frac{V}{r^2}$$

Putting $f'(r) = 0$ and restoring the value of $V = \pi r^2 h$ we have

$$2\pi r - \pi h = 0 \text{ whence}$$

$$h = 2r.$$

Hence it will take less material to make a can whose height equals the diameter of its base than one of any other cylindrical shape. We do not have to examine the second derivative as we know there is a maximum as there is only one critical value possible.

(3) Here is another example. It is required to measure a certain unknown magnitude x with precision. Suppose that n equally careful observations of the magnitude are made, giving the result $a_1, a_2, a_3 \ldots\ldots a^n$. The errors of these observations are evidently $x - a_1$, $x - a_2$, $x - a_3, \ldots\ldots x - a_n$, some of which are positive and some negative. It has been agreed that the most probable value of x is such that it renders the sum of the square of the errors, namely $(x - a_1)^2 + (x - a_2)^2 + (x - a_3)^2 + \ldots\ldots + (x - a_n)^2$, a minimum. Let us take the first derivative of this expression and put it equal to 0, then $(x - a_1) + (x - a_2) + (x - a_3) + \ldots\ldots + (x - a_n) = 0$. Simplifying we get $nx = a_1 + a_2 + a_3 + \ldots\ldots + a_n$. It will be found interesting to compare this with measurement in Section 10 of Chapter 1.

We can readily sketch Fig. 144 in Algebraic Geometry by means of the calculus, as shown here:

$$f(x) = 2x^3 - 5x^2 + 3x - 4$$

$$f'(x) = 6x^2 - 10x + 3$$

$$f''(x) = 12x - 10$$

$$''.392 = -, f''1.274 = +$$

Solving $6x^2 - 10x + 3 = 0$, critical values are .392 and 1.274.

Hence at $x = .392$ there is a maximum and at $x = 1.274$ a minimum, as in the figure.

10. Taylor's Theorem.—We have seen that numbers were written in powers of 10, and we have developed

many algebraic expressions in powers of x, but many trigonometric functions appear in different form. Calculus supplies a simple and general method, known as Taylor's theorem (see (1) below), for developing functions in power series. Let $\phi(x)$ be any function of x developable in the form. $\phi(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3 + \ldots$, where a, A, B, C, etc., are constants and the series is convergent. By successive differentiation $\phi'(x) = B + 2C(x-a) + 3D(x-a)^2 +$ $\phi''(x) = 2C + 2 \times 3D(x-a) + \ldots\ldots$ etc.

Now since these equations and the original from which they were derived are true for all values of x, they are true when $x = a$. They then become

$$\begin{aligned} \phi(a) &= A \\ \phi'(a) &= 1B \\ \phi''(a) &= 1.2C \\ \phi'''(a) &= 1.2.3D \end{aligned} \qquad \text{or} \qquad \begin{aligned} A &= \phi(a) \\ B &= \phi'(a) \\ C &= \frac{\phi''(a)}{2!} \\ D &= \frac{\phi'''(a)}{3!} \end{aligned}$$

Whence $\phi(x) = \phi(a) + \phi'(a)(x-a) + \phi''(a)\ \frac{(x-a)^2}{2!}$

$$+ \phi'''(a)\ \frac{(x-a)^3}{3!} + \ldots\ldots \quad (1).$$

If $a = 0$ Taylor's theorem becomes *Maclaurin's theorem*, as follows:

$$(2)\ \phi(x) = \phi(0) + \phi'(0)x + \frac{\phi''(0)x^2}{2!} + \frac{\phi'''(0)x^3}{3!} + \text{etc.}$$

When x in Taylor's theorem is replaced by $(a + x)$ we have

$$(3)\ \phi(a+x) = \phi(a) + \phi'(a)x + \frac{\phi''(a)x^2}{2!} + \frac{\phi'''(a)x^3}{3!} + \text{etc.}$$

Let us develop $\sin x$, assuming it developable, using Maclaurin's theorem.

$\phi(x) = \sin x \qquad \phi(0) = 0$, since $\sin 0° = 0$

$\phi'(x) = \cos x \qquad \phi'(0) = 1$, since $\cos 0° = 1$

$\phi''(x) = -\sin x \qquad \phi''(0) = 0$

$\phi'''(x) = -\cos x \qquad \phi'''(0) = -1$ Hence

$$\sin x = 0 + (1)x + \frac{(0)\,x^2}{2!} + \frac{(-1)\,x^3}{3!} + \ldots\ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots \text{ Similarly}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$$

(2) Let us expand $\sin (a + x)$, using form (3)

$\phi(a + x) = \sin (a + x) \qquad \phi(a) = \sin a$

$\phi'(a + x) = \cos (a + x) \qquad \phi'(a) = \cos a$

$\phi''(a + x) = -\sin (a + x) \qquad \phi''(a) = -\sin a$

$\phi'''(a + x) = -\cos (a + x) \qquad \phi'''(a) = -\cos a$

$$\sin (a + x) = \sin a + \cos ax + \frac{\sin ax^2}{2!} - \frac{\cos ax^3}{3!} + \frac{\sin ax^4}{4!} + \frac{\cos ax^5}{5!}$$

$$\text{(Factoring) } \sin (a + x) = \sin (a)\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{6!} + \ldots\ldots\right] + \cos a\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\right]$$

Whence $\sin (a + x) = \sin a \cos x + \cos a \sin x$ which agrees with trigonometry and gives us increased confidence in these new forms for sine and cosine.

Let us expand e^x, using Maclaurin's series.

$\phi(x) = e^x$ and by successive differentiation (Section 7 of this chapter) the successive derivatives are all e^x.

$\phi(0) = e^\circ = 1$, therefore all of the $\phi(0)$'s in Maclaurin's series $= 1$.

Hence $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$ (when $x = 1$ this is e).

(The binomial theorem is easily seen to be a special case of this great theorem.)

11. The Value of π.—The following is the development of arctan x by Maclaurin's formula:

$$\phi(x) = \arctan x \qquad \phi(0) = 0$$
$$\phi'(x) = \frac{1}{(1+x^2)} \qquad \phi'(0) = 1$$

The expression $\frac{1}{1+x^2}$ offers difficulties in differentiation. By division, when x is less than 1, $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots\ldots$ Hence

$$\phi''(x) = -2x + 4x^3 - 5x^5 + \ldots\ldots \qquad \phi''(0) = 0$$
$$\phi'''(x) = -2 + 3.4x^2 - 5.6x^4 + \ldots \qquad \phi'''(0) = -2$$
$$\phi'^{v}(x) = 2.3.4x - 4.5.6x^3 + \ldots \qquad \phi'^{v}(0) = 0$$
$$\phi^{v}(x) = 2.3.4 - 3.4.5.6x^2 + \ldots \qquad \phi^{v}(0) = 4!$$
$$\phi^{vi}(x) = -2.3.4.5.6x + \ldots \qquad \phi^{vi}(0) = 0$$
$$\phi^{vii}(x) = -2.3.4.5.6 + \ldots\ldots \qquad \phi^{vii}(0) = -6!$$

Hence arctan $(x) = 0 + x + 0 + \frac{-2x^3}{3!} + 0 + \frac{4!x^5}{5!} + 0 + \frac{-6!x^7}{7!} + \ldots = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots$

Our expression for arctan x now becomes:

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \ldots\ldots$$

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left[1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \ldots\ldots\right]$$

$$\pi = 2\sqrt{3}\left[1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} + \ldots\ldots\right]$$

In computing the value of π we will carry out our results to seven places of decimals and reject the last two places in the results as undependable. Note the arrangement of the calculation; how each term is obtained from the previous one, the positive and negative terms placed in different columns, added separately, combined and then multiplied by $2\sqrt{3}$.

	Positive	Negative
$3)\underline{1.0000000} \div 1 =$	1.0000000	
$3)\underline{.3333333} \div 3 =$		.1111111
$3)\underline{.1111111} \div 5 =$	.0222222	
$3)\underline{.0370370} \div 7 =$		.0052910
$3)\underline{.0123457} \div 9 =$	.0013717	
$3)\underline{.0041152} \div 11 =$		.0003741
$3)\underline{.0013717} \div 13 =$	.0001055	
$3)\underline{.0004572} \div 15 =$		.0000305
$3)\underline{.0001524} \div 17 =$	.0000090	
$3)\underline{.0000508} \div 19 =$		.0000027
$3)\underline{.0000169} \div 21 =$	.0000008	
$3)\underline{.0000056} \div 23 =$		.0000002
$3)\underline{.0000019} \div 25 =$	.0000001	
	1.0237093	.1168096
	.1168096	
	.9068997	
$2\sqrt{3} =$	3.464102	
	2.7206991	
	3627599	
	544139	
	36276	
	907	
	18	
$\pi =$	3.1415930	

12. Computation of Logarithms.—Taylor's series is very useful in the computation of logarithms, therefore let us obtain $\log (1 + x)$ by its means. (It is impossible to obtain $\log x$ in powers of x, for using Maclaurin's series, the only one applicable, $\phi'(x)$ would equal $\frac{1}{x}$ and $\phi'(0)$ would be infinite).

$$\phi(1 + x) = \log (1 + x) \qquad \phi(1) = \log (1) = 0$$

$$\phi'(1 + x) = \frac{1}{1+x} \qquad \phi'(1) = 1$$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 \ldots$ when x is less than 1.

Hence

$$\phi''(1+x) = -1 + 2x - 3x^2 + \ldots \quad \Big| \quad \phi''(1) = -1$$
$$\phi'''(1+x) = 2 - 2(3x) + \ldots \quad \Big| \quad \phi'''(1) = 2$$
$$\phi'^{v}(1+x) = -2.3 + \ldots \quad \Big| \quad \phi'^{v}(1) = -3!$$

Hence $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots$

Putting $-x$ for x, we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x5}{5} - \ldots\ldots$$

(In subtracting logarithms we divide numbers)

Subtracting, $\log \frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right]$

and is convergent when x is less than 1.

As the series converges very slowly and is otherwise inapplicable for computation, we will let $\frac{1+x}{1-x} = \frac{M}{N}$. Solving for x we get $x = \frac{M-N}{M+N}$. Evidently when M and N are positive, x is less than 1. Our series now becomes $\log \frac{M}{N} = \log M - \log N = 2\left[\frac{M-N}{M+N} + \frac{1}{3}\left(\frac{M-N}{M+N}\right)^3 + \frac{1}{5}\left(\frac{M-N}{M+N}\right)^5 + \ldots\right]$ which is a series convergent for all positive values for M and N and one in which it is always possible to choose M and N so as to make it converge rapidly.

Put $M = 2$; $N = 1$ $\left(\text{Note: } \log 1 = 0, \frac{M-N}{M+N} = \frac{1}{3}\right)$.

We get $\log 2 = 2\left[\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \ldots\ldots\right]$

The computation is as follows:

Putting $M = 3$; $N = 2$; the computation follows: $\frac{M-N=1}{M+N=5}$

3)2.0000000				5)2.0000000			
9) .6666667	÷	1 =	.6666667	5) .4000000	÷	1 =	.4000000
9) .0740741	÷	3 =	.0246914	5) .0800000			
9) .0082305	÷	5 =	.0016461	5) .0160000	÷	3 =	.0053333
9) .0009145	÷	7 =	.0001306	5) .0032000			
9) .0001016	÷	9 =	.0000113	5) .0006400	÷	5 =	.0001280
9) .0000113	÷	11 =	.0000010	5) .0001280			
.0000013	÷	13 =	.0000001	5) .0000256	÷	7 =	.0000036
			.6931472	5) .0000051			
$\log_e 2^2$ = .69315			2	.0000010	÷	9 =	.0000001
2 log 2 = log 4 =			1.3862944	log 2		=	.6931472
				log 3			1.0986122

13. Logarithms of Other Numbers.—It is only necessary to compute logarithms of prime numbers.

$$\log 5 = \log 4 + 2\left[\frac{1}{9} + \frac{1}{3.9^3} + \frac{1}{5.9^5} + \ldots.\right] = 1.60944$$

$$\log 7 = \log 6 + 2\left[\frac{1}{13} + \frac{1}{3.13^3} + \frac{1}{5.13^5} + \ldots.\right] = 1.94591$$

$$\log 10 = \log 2 + \log 5 = 2.30259$$

All of the above are natural logarithms to base $e = 2.7182818$. The logarithms in our tables are to base 10. To get them, let $n = \log_{10} N$ or $10^n = N$ and take logarithms to base e. We get $n \log_e 10 = \log_e N$, whence

$$n = \frac{\log_e N}{\log_e 10}, \text{ whence } \log_{10} N = \frac{\log_e N}{2.30259} = .43429 \log_e N.$$

Hence $.43429 \times .69315 = .3010 = \log_{10} 2$.

REVIEW.

1. What is a derivative? Define the derivative of a fraction.
2. Distinguish between derivatives and differentials.
3. Describe briefly the meaning of maxima and minima.
4. How are functions developed in power series?
5. Illustrate the use of Taylor's series in computing logarithms.

CHAPTER XV

INTEGRAL CALCULUS

1. Anti-Differentiation.—We have become familiar with the mutually inverse operations of addition and subtraction, multiplication and division, involution and evolution. Integration and differentiation are again inverses. It took the world at least 1800 years to produce a Newton able to say that anti-differentiation was the same as integration invented by Archimedes.

We started out the Differential Calculus with $f(x) = x^2$ and obtained $f(x)\,dx = 2x\,dx$. We will now say inversely

$\int 2x\,dx = x^2$, in which the integral sign $\int$ is an old-fashioned "S" from "sum".

If $f(x) = x^2 + C$, then $f(x) = 2x\,dx$ and $\int 2x\,dx = x^2 + C$. The differential is perfectly definite but the same integral may express any number of values differing by a constant. Our data are satisfied by any one of the parabolas represented by $y = x^2 + C$, but to distinguish which one we must have some additional information as, for instance, a point, say (1,2) on the parabola. Substituting we get $2 = 1 + C$, hence $C = 1$ and our parabola is $y = x^2 + 1$. This is as in Fig. 135 moved up one unit.

2. The Formula for Falling Bodies.—It is a matter of physics that the acceleration of gravity is a constant, generally called g and equal to approximately 32 feet per second. This means that the change of the velocity

in respect to the time t equals g, that is, $\frac{dv}{dt} = g$. In differential notation this is $dv = g\,dt$. Hence $\int dv = \int g\,dt = g\int dt$ (g being a constant). As the integral and differential signs are inverse they cancel each other and we get $v = gt + C$.

If $t = 0$, when $v = 0$, C is 0 also, that is $v = gt$. But v is the change of the distances in respect to the time; hence $\frac{ds}{dt} = gt$ in differential form $ds = gt\,dt$ and integrating $\int ds = g\int t\,dt$. Hence $s = \frac{gt^2}{2} + C$. If $t = 0$, when $s = 0$, C is 0 also and $s = \frac{1}{2}gt^2$, the formula in physics where a body starts from rest.

Notice that $t\,dt$ was not a perfect differential and that $2t\,dt$ is the differential of t^2.

For the first time in mathematics the summation form is the indirect and not the direct operation. It is always necessary to ask, as Archimedes should have done, what was differentiated in order that we may integrate. We should write $\int t\,dt = \frac{1}{2}\int 2t\,dt = \frac{1}{2}t^2 + C$. This thought is extremely fundamental.

The only way we arrive at the primitive of a given function is thru our previous knowledge of what function differentiated will yield the given function. Formula (6) of Differential Calculus written in a differential form is

$$d(u^n) = nu^{n-1}du.$$

We lowered the exponent by 1 and multiplied by the old exponent. In integral calculus we do the reverse of this: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ and we might say that x^n must have been found by some one in differentiating x^{n+1}. Now $d(x^{n+1}) = (n+1)x^n$; hence as $n+1$ is a constant and will go as readily inside as out of the inte-

gral sign, as it did on either side of the differential sign, we write

$$\int x^n dx = \frac{1}{n+1} \int (n+1) x^n dx.$$

We know by our operation of differentiating x^{n+1} that the expression under the integral sign is that differential. Its integral must be what we differentiated, hence $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

$\int (ax^3 + b)^n x^2 dx = \frac{1}{3a} \int (ax^3 + b)^n (3ax^2 dx)$, which

is now of the form $\frac{1}{3a} \int u^n du = \frac{u^n + 1}{3a(n+1)} + C$, whence

$$\frac{1}{3a} \int (ax^3 + b)^n (3ax^2 dx) = \frac{(ax^3 + b)^{n+1}}{3a(n+1)} + C.$$

Let us examine $\int \left(1 + \frac{1}{ax^2 + b}\right) \frac{x dx}{ax^2 + b}$.

In this the differential of the quantity in parenthesis is $\frac{-2ax\,dx}{(ax^2 + b)^2}$, which is not the same as the quantity outside of the parenthesis. Multiplying out the expression, we get $\int \frac{x dx}{ax^2 + b} + \int \frac{x dx}{(ax^2 + b)^2}$ where the last expression, except for constant $(-2a)$, is the same as the differential of $\frac{1}{ax^2 + b}$. We can therefore integrate the second part but the first part will give trouble unless we notice that the differential of the denominator is $2axdx$. If I represents the original integral we now have $I = \frac{1}{2a} \int \frac{2ax\,dx}{ax^2 + b} - \frac{1}{2a} \int \frac{-2ax\,dx}{(ax^2 + b)^2} = \frac{1}{2a} \log (ax^2 + b)$

$$- \frac{1}{2a} \left(\frac{1}{ax^2 + b}\right) + C = \frac{1}{2a} \left[\log (ax^2 + b) - \frac{1}{ax^2 + b} + C.\right]$$

The reader must be always on the watch for the *numerator as the differential of the denominator*, which is the reverse of (12′) or (12). We could not integrate as

a power $\int \frac{du}{u} = \int u^{-1}du = \frac{u^0}{0}$ for division by 0 has no meaning. The -1 power is the dangerous one and we should be always on our guard and remember that $\int \frac{du}{u} = \log u + C = \log u + \log K = \log Ku$.

3. An Example in Interest.—If P dollars be drawing compound interest at r per cent; in any time, Δt, the interest is $\frac{r}{100} P\Delta t$, which equals the change in P. If the interest is added on continuously we have

$dP = \frac{r}{100} P\,dt$, that is, the derivative is proportional to the sum itself and we would expect an exponential form. Separating variables we get $\frac{dP}{P} = \frac{r\,dt}{100}$, whence $\log P = \frac{rt}{100} + C$. Now if $P = 1$ when $T = 0$, that is, if we start out with one dollar at the beginning, C will equal 0 and $\log P = \frac{rt}{100}$, whence $P = e^{\frac{rt}{100}}$. If the initial sum had been a we would have had $\log a = C$, whence $\log P - \log a = \frac{rt}{100}$, hence $\log \left(\frac{P}{a}\right) = \frac{rt}{100}$, hence $\frac{P}{a} = e^{\frac{rt}{100}}$ and $P = ae^{\frac{rt}{100}}$.

We might ask ourselves how long will it take a sum of money to *double* itself at 4% compound interest in an insurance company or any other business investing its money continuously. Here $P = 2$, $a = 1$ and $r = 4$ and $\frac{4t}{100} = \log_e 2 = .69315$, whence $4t = 69.315$ and $t = 17.429$ years. If $r = 6$ we have $6t = 69.315$ and $t = 11.553$ years, as in a building and loan association.

4. Area Under a Curve.—Suppose we have a plot of $y = 3x^2 + 5$, where we know some initial area $z = OBAC$, Fig. 149. If we give x an increment $\Delta x = AE$

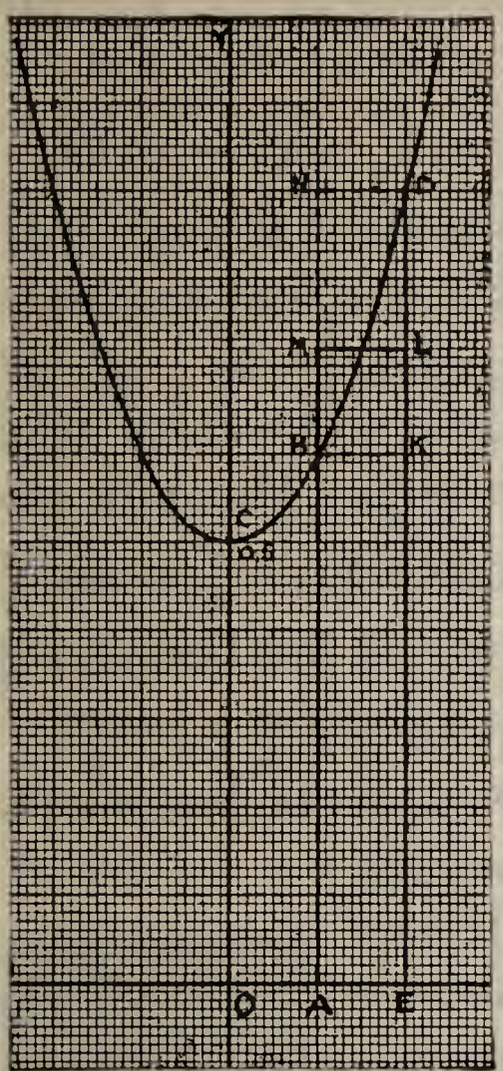

Fig. 149.

and consider the increment $\Delta z = AEDB$ of the area z, equal to a rectangle $AELM$ greater than the rectangle $AEKB$ and less than the rectangle $AEDH$ whose altitude AL is equal to the y of some point of the curve between B and D. Evidently as Δx approaches 0 this y differs from either AB or ED by less than they differ from each other. It can, therefore, in general be stated that the increment of the area $AEDB$ is equal to the rectangle $AELM$ by means of the symbolism

$$dz = y\,dx$$

where y is the ordinate of the curve and dx is a small Δx. Hence $z = \int y\ dx$. In our case $z = \int (3x^2 + 5)dx$, whence $z = x^3 + 5x + C$.

It now remains to find the value of the constant C. If we wish to determine the area from the y axis, where $x = 0$, $z = 0$ and hence $C = 0$. When $x = 3$, $z = 42$, hence the area between the curve, the axes of coordinates and an ordinate at $x = 3$, is 42. This is symbolically written as follows:

$$\int_0^3 (3x^2 + 5)dx = \Big[x^3 + 5x\Big]_0^3 = 42.$$ We might as readily have found

$$\int_3^5 (3x^2 + 5)dx = \Big[x^3 + 5x\Big]_3^5 = 150 - 42 = 108.$$

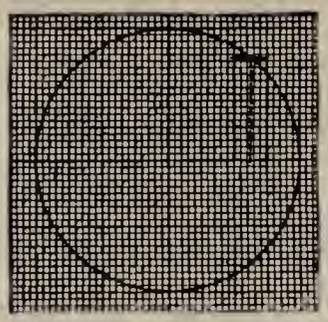

Fig. 150.

5. Area of Circle.—Let us find area of circle $x^2 + y^2 = a^2$, Fig. 150.

$$\frac{A}{4} = \int_0^a y\,dx = \int_0^a \sqrt{a^2 - x^2}\,dx.$$

We will use formula (14) page 263. If we substitute the limits a and 0 in this expression, $\frac{x}{2}\sqrt{a^2 - x^2}$ will equal 0 in both cases. The radical will vanish when $x = a$ and $\frac{x}{2}$ will vanish when $x = 0$. The only part that concerns us in this particular problem is $\frac{a^2}{2}\arcsin\frac{x}{a}$. The remaining part vanishes for $x = 0$ but becomes $\frac{a^2}{2}$ arcsin 1 for $x = a$, which reduces to $\frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$.

The area of the circle, Fig. 151, could be obtained by means of polar instead of rectangular coordinates. In trigonometry the area of a triangle is $\frac{1}{2}\,ab \sin c$ and we learned in the same branch of mathematics that $\frac{\sin\theta}{\theta} = 1$. The elemental triangle whose included angle is $\Delta\theta$ in the figure becomes in the limit $\frac{1}{2}r^2 d\theta$. Summing up such triangles, the area $= 4\left(\frac{1}{2}\right) r^2 \int_0^{\frac{\pi}{2}} d\theta = \pi r^2$. Here r is a constant.

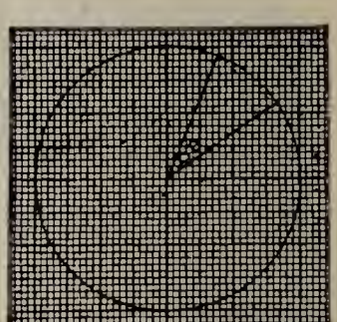

Fig. 151.

6. Area Between Curves.—If the area between two curves is wanted, we have to find by simultaneous equations where the curves intersect. Find the area between the two parabolas, $y^2 = 4 + x$ and $y^2 = 4 - x$, Fig. 152. Solving these equations simultaneously, since y^2 is in each, $4 + x = 4 - x$ and $x = 0$, whence $y = \pm 2$.

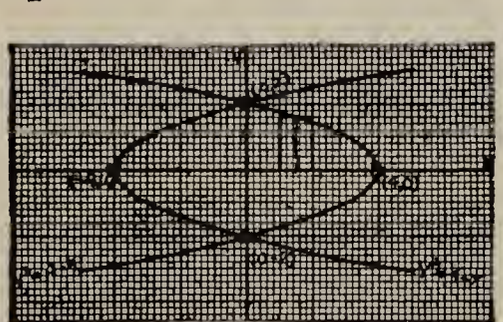

Fig. 152.

Taking advantage of the fact of symmetry we find only the upper righthand quarter, which is obtained from the equation $y^2 = 4 - x$. We therefore say

$$\int_0^4 y\,dx = \int_0^4 (4-x)^{\frac{1}{2}}\,dx = -\left[\tfrac{2}{3}(4-x)^{\frac{3}{2}}\right]_0^4 = \tfrac{16}{3}.$$

We could also have said

$$\int_0^2 x dy = \int_0^2 (4-y^2)dy = \left[4y - \tfrac{y^3}{3}\right]_0^2 = \tfrac{16}{3}.$$

7. Length of Curve.—We learn in trigonometry that the projection of a line l is $l \cos\theta$ where θ is the angle which the line makes with the x axis. If a is the projection, evidently $l = a \sec\theta$. In geometry $\frac{dy}{dx}$ is the slope of the curve and since $\sec\theta = \sqrt{1 + \tan^2\theta}$, then $\sec\theta = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ and if dx is the projection of an element of the curve ds, then $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$. This is evidently identical with $\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$.

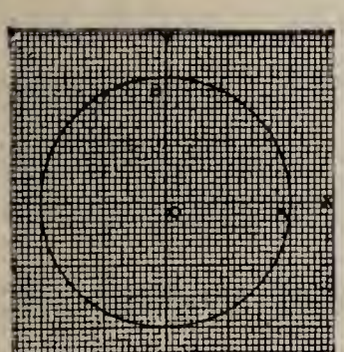

Fig. 153.

Find the length of the circle $x^2 + y^2 = r^2$, Fig. 153.

Differentiating $\frac{dy}{dx} = -\frac{x}{y}$

whence since $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$\text{arc } AB = \int_0^r \left[1 + \frac{x^2}{y^2}\right]^{\frac{1}{2}} dx$$

$$= \int_0^r \left[\frac{y^2+x^2}{y^2}\right]^{\frac{1}{2}} dx = \int_0^r \left[\frac{r^2}{r^2-x^2}\right]^{\frac{1}{2}} dx.$$

(Substituting $y^2 = r^2 - x^2$ from the equation of the circle in order to get everything in terms of x.)

$$\text{Therefore arc } AB = r\int_0^r \frac{dx}{\sqrt{r^2-x^2}} = \left[r \text{ arc sin } \frac{x}{r}\right]_0^r = \frac{\pi r}{2}.$$

Hence the total length equals $2\pi r$. Ans.

8. Volumes of Solids of Revolution.—The sphere can be obtained by revolving a circle about its diameter, as

can be readily seen by turning an egg beater fast. The sphere could be made up of a number of thin circular plates whose sizes vary from the original circle to a point. Each one of these plates is equivalent to a little cylinder, having a radius y and a thickness dx, where x is the axis of revolution. Each cylinder will have the volume $\pi y^2\,dx$ and the volume of the sphere will be $2\pi \int_0^a y^2 dx$. The 2 comes from the fact that the center is at the origin. The equation of the circle is $x^2 + y^2 = a^2$, whence volume $= 2\pi \int_0^a (a^2 - x^2)dx$, since $y^2 = a^2 - x^2$. Integrating, we get $2\pi \left[a^2x - \frac{x^3}{3}\right]_0^a = 2\pi \left(\frac{2a^3}{3}\right) = \frac{4}{3}\pi a^3$.

9. Areas of Surfaces of Revolution.—If the sphere in the preceding example were to be cut into thin slices, each slice would be a little frustum of a cone. The small element of surface has for its slant height ds, for the original problem under volumes would have shown steps like the pyramids of Egypt. These steps would have been dy in width and dx in height, and the length from one projection to the next would be $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$. And since the curved surface of a frustum of a cone $= 2\pi al$, where l is the slant height, the area of this surface will be $4\pi \int_0^a y \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$ = (as shown above) $4\pi a \int_0^a dx = 4\pi a^2$.

Notice that the y's cancel and the numerator reduces to the radius.

10. Formulas of Integration.—The formulas of integration most frequently used are summarized herewith:

(1) $\int a\,dx = ax + C$

(2) $\int x^n dx = \frac{x^n+1}{n+1} + C$ (when n is not $= -1$),

(3) $\int \frac{dx}{2\sqrt{x}} = \sqrt{x} + C$

(4) $\int x^{-1} dx = \int \frac{dx}{x} = \log x + C$

(5) $\int a^x dx = \frac{a}{\log a} + C$

(6) $\int e^x dx = e^x + C$

(7) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \text{ arc tan } \frac{x}{a} + C$

(8) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$

(9) $\int \frac{dx}{a^2 - x^2} = \text{arc sin } x + C$

(10) $\int \sin x\,dx = -\cos x + C$

(11) $\int \cos x\,dx = \sin x + C$

(12) $\int \sec^2 x\,dx = \tan x + C$

(13) $\int \sec x \tan x = \sec x$

(14) $\int \sqrt{a^2 - x^2}\,dx = \frac{a^2}{2} \text{ arc sin } \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}$

(15) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log (x + \sqrt{x^2 \pm a^2}) + C$

(Consult a table of integrals for any integral not mentioned here. Works on integral calculus are necessarily lengthy on account of the treatment of special integrals).

REVIEW.

1. How is integration related to differentiation?
2. Show how interest problems may be solved by integration.
3. Find the area of the circle in Section 5 from $\frac{A}{4} = \int_0^a x dy$.
4. Find the area of the curve $y = x^2$, Fig. 135, below the line $y = 10$. Count the squares inside the area and test your work.
5. Find the volume generated by revolving the portion of the curve determined in Question 4 about the y axis.

PHYSICS

EDITOR'S PREFACE TO PHYSICS

The continually widening field of application of scientific knowledge in the useful arts has made it necessary for all who desire to understand modern mechanical devices to be familiar with the main principles of the sciences. The study of Physics includes the great bulk of the scientific principles which have been most directly applied in the arts. In this Text an attempt has been made to present the principles of Physics in a simple and orderly fashion so that the reader may get clear and accurate ideas concerning mechanics, heat, sound, light and electricity which, taken together, constitute a complete presentation of the elements of the subject. The order in which these subdivisions are placed is of little importance except that mechanics should come first since this subject contains the essential principles of every branch of Physics.

Altho Physics is distinctly a mathematical science, yet in order to render the Text as interesting as possible the mathematical mode of treatment has been avoided wherever this could be done. Nevertheless the reader will find it interesting as well as profitable to render himself expert in the solution of problems based on the formulae developed in the text. It is also very desirable that as many as possible of the simple experiments described in the Text should be repeated by the reader, since personal experimentation is a wonderful help toward mastering the ideas involved. A single experiment thoughtfully performed has often been found more valuable in this respect than many pages of explanation.

It is hoped that a study of these pages will give the reader greater ability to interpret the phenomena of nature which may come under his observation and to understand the operation of the marvelous mechanics of modern inventions.

CHARLES B. BAZZONI.

November, 1920.

TABLE OF CONTENTS

INTRODUCTION

SECTION		PAGE
1.	The Purpose of Physics	277
2.	Physics Defined	278
3.	Conservation of Matter and Energy	280
4.	Measurements	281
5.	The Metric System	282
6.	How Facts are Determined	283
7.	Direct and Inverse Variations	284
8.	General Classifications	285

CHAPTER I

THE MECHANICS OF LIQUIDS

SECTION		PAGE
1.	Hydrostatics	287
2.	Pascal's Law	288
3.	Depth and Pressure	290
4.	Archimedes' Principle	294
5.	Determinations of Density	295
6.	Specific Gravity	296
7.	Densities of Solids and Liquids	296
8.	Law of Flotation	298

CHAPTER II

THE MECHANICS OF GASES

SECTION		PAGE
1.	Atmospheric Pressure	300
2.	The Barometer	302
3.	Weather Predictions	304
4.	Internal and External Pressure	305

5. Boyle's Law of Pressure and Density..... 306
6. Height of the Atmosphere..... 307
7. The Balloon..... 309
8. The Siphon..... 310
9. The Air Pump..... 311
10. The Lift Pump for Water..... 312
11. The Force Pump for Water..... 312
12. The Diving Bell..... 313
13. The Air Brake..... 314

CHAPTER III

THE MOLECULAR THEORY OF MATTER

SECTION PAGE
1. Molecules 315
2. Molecular Motion..... 316
3. Effect of Heat on Rate of Motion..... 319
4. Evaporation and Condensation..... 319
5. Clouds, Rain, Dew, etc..... 322
6. Evaporation and Temperature..... 323
7. Determining the Relative Humidity..... 324
8. Evaporation in a Vacuum..... 325
9. Molecular Forces in Liquids..... 326
10. Cohesion and Adhesion..... 328
11. Capillary Elevation and Depression..... 330
12. Molecular Forces in Solids..... 331
13. Hooke's Law of Elasticity..... 333

CHAPTER IV

COMPOSITION AND RESOLUTION OF FORCES

SECTION PAGE
1. Force Units..... 334
2. Measurement and Comparison of Forces..... 335
3. Resultants and Equilibrants..... 336
4. Components 338
5. Force Actions: Horizontal Plane..... 339
6. Force Actions: Inclined Plane..... 340

7. Newton's Three Laws 341
8. Inertia 341
9. Momentum 342
10. Action and Reaction 345
11. Attraction of Bodies 347
12. Accelerated Motion: Falling Bodies 348
13. Equilibrium 351

CHAPTER V

WORK, ENERGY AND MACHINES

SECTION PAGE
1. Work 353
2. Power 355
3. Energy: Potential and Kinetic 356
4. The Six Simple Machines 361
5. The Lever 362
6. The Wheel and Axle 363
7. The Pulley 364
8. The Inclined Plane 365
9. The Screw 366
10. Friction 366

CHAPTER VI

HEAT

SECTION PAGE
1. Heat and Energy 368
2. Temperature 369
3. The Thermometer 370
4. Centigrade and Fahrenheit 372
5. Measuring High and Low Temperatures 374
6. Transference of Heat: Conduction 376
7. Convection 377
8. Radiation 379
9. Effects of Heat on Matter 381
10. Coefficients of Expansion—Solids 382
11. Expansion of Liquids 383
12. Expansion of Gases 385

13. Law of Charles and Gay-Lussac..... 386
14. The Absolute Scale of Temperature..... 387

CHAPTER VII

QUANTITY OF HEAT, CHANGE OF STATE AND HEAT ENGINES

SECTION PAGE
1. Quantity of Heat..... 390
2. Specific Heat..... 392
3. Calorimeter 394
4. Changes in State..... 395
5. Heat of Fusion..... 397
6. Melting Point..... 398
7. Vaporization 400
8. Boiling Point of Water..... 401
9. Absorption and Liberation of Heat..... 403
10. Reciprocating Steam Engine..... 405
11. Steam Turbine..... 406
12. Gas Engine..... 407

CHAPTER VIII

SOUND—ITS NATURE AND PROPAGATION

SECTION PAGE
1. Common Source..... 410
2. Transmission of Sound..... 411
3. The Tuning Fork..... 413
4. Speed of Sound Waves..... 415
5. Reflection of Sound..... 416
6. Law of Inverse Squares..... 418
7. Pitch 420
8. Musical Sounds and "Noise"..... 422

CHAPTER IX

MUSICAL SCALES AND INSTRUMENTS

SECTION PAGE
1. The Musical Scale..... 425
2. Intervals and Chords..... 426

3. Resonance 430
4. Beats 433
5. Organ Pipes—Open and Closed 435
6. Wind Instruments 437
7. Stringed Instruments 438
8. Quality and Overtones 441
9. The Phonograph 443

CHAPTER X

LIGHT—ITS NATURE AND PROPAGATION

SECTION PAGE
1. Nature of Light 444
2. Propagation of Light 445
3. Reflection 447
4. Shadows 448
5. The Wave Theory of Light 451
6. Phenomenon of Interference 452
7. Refraction 454
8. The Critical Angle 457
9. Intensity and Distance 458
10. The Photometer 460

CHAPTER XI

THE FORMATION OF IMAGES AND OPTICAL INSTRUMENTS

SECTION PAGE
1. Virtual and Real Images 462
2. Image Formation by Reflection 463
3. Convex Spherical Mirror 465
4. Concave Spherical Mirror 466
5. Image Formation by Refraction 468
6. Construction of Images: (a) Convex Lenses 470
(b) Concave Lenses 472
7. Summary on Images Formed by Mirrors and Lenses .. 473
8. Optical Instruments—the Camera 473
9. Magnifying Glass 475

10. Telescope and Compound Microscope..... 475
11. Opera Glass..... 477
12. Prism Binocular..... 477

CHAPTER XII

COLOR AND WAVE LENGTH OF LIGHT

SECTION PAGE
1. Dispersion of Light..... 478
2. Colors of Opaque Objects..... 479
3. Complementary Colors..... 480
4. Mixed Colors and Mixed Pigments..... 481
5. The Spectroscope..... 482
6. The Rainbow..... 484
7. Chromatic Aberration..... 485
8. Long and Short Waves—the Radiation Spectrum... 486

CHAPTER XIII

STATIC ELECTRICITY

SECTION PAGE
1. Electrification 489
2. Positive and Negative Electricity..... 490
3. Electrons 492
4. Frictional Electricity..... 493
5. Distribution of Electrons..... 494
6. Potential 495
7. The Voltaic Cell..... 496

CHAPTER XIV

MAGNETISM—OHM'S LAW

SECTION PAGE
1. Magnets 498
2. Magnetic Field of Force..... 500
3. Terrestrial Magnetism..... 501
4. Oersted's Discovery..... 502

5. Galvanometers 503
6. Electrometers 504
7. Electro-Motive Force 506
8. Voltmeters 507
9. Conductivity and Resistivity 507
10. Ohm's Law 509
11. Series and Parallel Connections 511
12. Shunts 512

CHAPTER XV

ELECTRIC CURRENTS—THEIR EFFECTS

SECTION PAGE
1. Heat Effects: Joule's Law 513
2. Electric Heaters 515
3. Filament and Arc Lamps 515
4. The Cooper-Hewitt Lamp 516
5. Electrolysis 517
6. Polarization: Galvanic Cells 519
7. The Storage Battery 520
8. The Solenoid 521
9. The Electromagnet 521
10. The Electric Telegraph 523
11. Induced Currents 524
12. The Induction Coil 526
13. The Dynamo 527
14. Alternating Current: the Transformer 529
15. Direct Current 530
16. Direct Current Motor 531
17. The Telephone 531

PHYSICS

INTRODUCTION

1. The Purpose of Physics.—Physics is concerned very largely with the explanation of phenomena familiar to us all. We know, for example, that a stone will fall if released in mid-air, that a breeze exerts a force on the sail of a boat and that a kite will rise in a wind. We know that iron objects sink in water and that wooden ones float; that electricity produces light, heat and power and that sunlight shining thru a cloud will sometimes form a rainbow in the sky. We are all acquainted with these and similar phenomena from our ordinary experience. It is probable however that our knowledge of these matters is superficial in that it is not so accurately formulated as to permit us definitely to calculate the magnitude of the effects to be expected from any given causes. It is the purpose of physics so to systematize and correlate the accumulated facts resulting from the observation of natural processes that we may deduce therefrom generalizations or laws which, when expressed in mathematical form, will serve as the means for calculating the exact results to be expected from causes of a given magnitude.

Lord Kelvin, one of the most famous physicists of the nineteenth century, said:

> When you can measure what you are speaking about and express it in numbers, you know something about it, and when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may

be the beginning of knowledge, but you have scarcely in your thoughts advanced to the stage of a science.

This mathematical aspect of physics is the one which appeals to the engineer and to the scientist. Since most of the material advancements and many of the esthetic improvements of modern civilized life are developments of pure science or of physics applied in engineering, a general knowledge of physics is so essential to a proper understanding of the world in which we live that each of us, whether mathematically inclined or not, can pleasurably and profitably devote a certain time to the mastery of the fundamentals of this subject. In our Text the principles of physics are presented with a minimum of mathematics and in such a way that the reader may get not only the fundamental laws of the subject with the classified facts on which those laws are based, but also some conception of that scientific method of acquiring and interpreting facts which has led to so many marvelous discoveries in the past and is certain to lead to still more striking developments in the future.

2. Physics Defined.—Physics is ordinarily defined as that branch of science which treats of matter and energy —or as that branch of science which deals with the action of force on matter. It is necessary, therefore, in the first place to get some idea of the exact meaning of these terms—matter, force and energy. Matter is defined as that which occupies space and has weight—it therefore comprises all material objects in the universe in whatever state—gaseous, liquid or solid. The quantity of matter in a body is called the *mass* of that body. There is a mutual attraction in the nature of a pull or tension between all masses. The extent or magnitude of this

pull we shall study later. The earth's pull on any body due to mutual attraction is known as the *weight* of the body. The weight of a body is proportional to its mass. If one body has twice the mass of another, it will have at any one place twice the weight of the other. The quantity of matter in a cubic unit of a body determines the *density* of that body. Thus iron, having greater density than cork, it follows that a cubic meter of iron has greater weight than a cubic meter of cork. The density may consequently be defined as the *mass per unit volume* of the body.

Matter in all its forms is granular in structure. The granules are extremely small, however,—much smaller than the smallest particles that can be perceived with even the most powerful microscopes. These tiny granules are called *molecules.* The molecules are known to be about 0.00000003 centimeters in diameter. Moreover, each kind of matter has its own kind of molecule. For example, the molecules of salt, sugar, water, paper and ink are all different in kind. Molecules can be subdivided by certain processes into still smaller particles called *atoms.* There are known to be in the universe only a limited number of kinds of atoms—about eighty kinds all told. These different kinds of atoms taken separately constitute the eighty "chemical elements" of which gold, tin, lead, sulphur, oxygen, nitrogen and hydrogen may serve us as examples. These atoms cannot be subdivided by any ordinary agencies and were, in fact, thought to be indivisible until within a few years. It is now, however, fairly well established that atoms contain much smaller particles, all of one kind called *electrons.* These electrons are about $\frac{1}{2000}$ of the mass of an atom. This subject will be discussed at length later.

Force shows itself as a compression or tension which acting between two bodies tends to change their state of motion relative to one another. Whenever a body is seen to move, we know that a force is acting on it or has been acting on it. It is not possible for a new motion to be produced or for an existing motion to be changed in any way except thru the action of a force on the moving body. If the body acted upon by the force is rigidly supported so that it cannot move, it will nevertheless *tend* to move, the tendency being neutralized by a strain in the supporting members which is exactly equal to the acting force. It is well to keep clearly in mind that forces act invariably *between* two bodies, pushing or pulling on the one body equally as on the other.

Energy is a quality acquired by matter as a result of the action of force on it. The acting force as above explained moves the body or distorts it and so puts it into a position or condition to exert force and to do work itself. For example, if a force acts on a body and lifts it to a certain height, the body in falling from that height to its original level can do work on some mechanism—the stored up work in the body in its elevated position is the *energy* of the body at that moment. We may define energy as the ability to do work, remembering that no body can possess energy save as a result of the previous action upon it of some force.

3. Conservation of Matter and Energy.—It will be well to state at this point what may be called the fundamental theorem of physical science—the doctrine of the conservation of matter and energy. Matter as well as energy may be transformed from one state to another, but neither matter nor energy can be either created or destroyed. The sum total of matter in the universe is a

constant—or fixed quantity—as is also the sum total of energy. For example, when water is boiled away, the quantity of matter in the steam is exactly equal to that which was previously present in the water. Also when the body referred to in the preceding paragraph is lifted to a height and then allowed to fall, the energy transformed on the way down in working the mechanism or converted into heat in friction on impact at the bottom, must have exactly equalled and can never have exceeded the energy used in raising the body to the height in the first place. This doctrine is of universal application and is of the most fundamental importance. We shall have occasion to refer to this principle repeatedly and to explain it in various connections.

4. Measurements.—In physics we are usually concerned with studies of cause and effect. A phenomenon is observed to depend on certain variable factors. For example, the extension of a spiral spring is seen to depend on the magnitude of the weights suspended at the bottom. In order to determine the law of the phenomenon, it is necessary to take certain definite measurements linking up cause and effect. We might, for instance, suspend different weights from the spiral spring and measure the extension produced by each one. In this way we could readily establish a relation which would enable us to predict how much any suggested weight would extend the spring. The practical problems involved in this simple investigation are merely problems of measurement. All problems in physics are, speaking generally, problems of measurement. But before entering into a study of these problems, it is well to become familiar with the systems of units which are made use of in the measurements.

The measurements involved in the simple experiment described in the foregoing are of two quantities—(1) length and (2) mass or weight. It will be found as we progress further that all the measurement which we may find it necessary to take, of however complicated a character can, if we wish, be finally reduced to measurements of only three quantities—length, mass and time. These quantities are referred to as " fundamental quantities " and in any system of measurement the units of these quantities are known as the three fundamental units. For instance, in the ordinary so-called English Gravitational System of Units—the unit of length is the foot; of mass, the pound; and of time, the second. The system is therefore called the F. P. S., or foot-pound-second system. In the French or metric system of units—the unit of length is the centimeter; of mass, the gram; and of time, the second, and the system is called the C. G. S., or centimeter-gram-second system. We shall have occasion in this book to use both the F. P. S. and the C. G. S. systems, as well as a third system which will be described in due time.

5. The Metric System.—In pure scientific work, however, the metric system is in universal use, mainly because it is very much easier to work with than any other system. It is desirable therefore in studying physics that we get into the habit of thinking in terms of metric units. The system was originated by the French in 1793 and is built up on a measurement of the quarter circumference of the earth from the equator to the north pole. The meter was taken to be one ten-millionth part of this distance. However, a slight error was made in the calculations so that the meter as finally marked off was not exactly the length intended. We must there-

fore define the meter as being the distance between two scratches on a certain platinum-iridium bar kept in Paris. The centimeter is $\frac{1}{100}$ of the meter—the millimeter is $\frac{1}{1000}$ of a meter. One thousand meters form a kilometer.

Units of surface are based on the square centimeter (cm.^{2}). Units of volume are based on the cubic centimeter (cm.^{3}), that is, on a cube with an edge 1 cm. long. One thousand cubic centimeters constitute the unit of liquid measure, called the liter.

Units of mass are based on the gram, which is the mass of 1 cm.^{3} of pure water at 4° Centigrade, the temperature at which water is most dense—that is to say contains the most mass per unit volume.

The unit of force is the earth's pull on a gram of matter. This pull is called the gram of force. It is convenient to have in mind the following relations for interchanging English and metric units:

1 meter	= 1.1 yards=39.37 inches.
1 inch	= 2.54 cm.
1 kilogram (1000 grams)	= 2.2 pounds.
1 pound (avoirdupois)	= 454 grams.
1 liter	= 1.0567 quarts.
1 quart	= 0.9463 liters.

These relations are accurate enough for use in all ordinary calculations.

6. How Facts are Determined.—We are now in a position to consider the steps of the scientific method for determining accurate facts. If it is desired to study any particular phenomenon, as for example, the bending of a wooden lath, clamped at one end, under loads applied at the free end, we first make an experimental

set-up and determine from observation what factors affect the phenomenon, in this case, the bending. We can easily find that for a given material the amount of bending is dependent on the load applied—on the breadth of the lath—on its thickness and on its length. We must then vary these factors one at a time keeping the other conditions constant and make a series of measurements of the amount of bending corresponding to definite values of the quantity being varied. From a study of the tabulated results we can see how the bending varies with each of the factors and when we have expressed the ratio of variation mathematically and have collected all the variations into a single formula, we have a definite generalization which we can dignify by the name of a law—the " law of flexure of flat beams of wood supported at one end."

It is well for us to note carefully the exact meaning here given the word "law." In this connection we can quote Rowland and Ames—two well-known American physicists:

> A law of nature is a statement of our belief concerning certain phenomena; it is suggested by a number of observations and measurements and is, in fact, a generalization of these. It is shown to be in accord with all observations, to within the range of error inherent in the experimental instruments used, but can never be perfectly verified.

7. Direct and Inverse Variations.— Our study of the flexure of the lath would bring out two kinds of variation. We would find that the flexure would vary with the applied load—doubling the load would double the flexure. If f represents the amount of flexure, and w the load applied, this fact can be expressed mathematically as follows: $f \propto w$ where the symbol $\propto$ represents

the phrase "varies with." Under these conditions we see that $\frac{f}{w}$ is a constant number and, representing this constant by k, $\frac{f}{w}=k$. This is an example of *direct variation.* Whenever the quotient of the corresponding values of two variables is a constant, we know that the variables vary directly with one another.

In determining the effect of changes in breadth on the flexure of the lath, keeping the load constant, we would find that doubling the breadth would reduce the flexure one-half. If b represents the breadth it will be seen that in this case the product of b and f will be a constant number or

$$b \times f = k \text{ which can be written}$$
$$f = k \times \frac{1}{b} \text{ or } f \propto \frac{1}{b}.$$

This is an example of an *inverse variation*—f is said to vary inversely with b. Whenever the product of the corresponding values of two variables is a constant, the variables are known to vary inversely with one another. As we continue our work in physics, we shall come upon a great many examples of direct and inverse variations.

8. General Classifications.— The subject-matter of physics is classified, as can be seen on looking thru the Table of Contents of this Text, into the following sub-divisions: Mechanics, Heat, Sound, Light and Electricity. We are now ready to take up a consideration of the first of these divisions—Mechanics, which is concerned with the action of forces on bodies of matter. It comprises two sub-divisions, (a) Statics, (b) Dynamics. Statics treats of the action of *completely balanced* systems of forces on bodies. When the force system is balanced no motion can result, altho strains are of course set up in the body. If, for example, you

hang your overcoat, weighing 7 pounds, on a wardrobe hook, the hook resists the downward pull of the coat with a force of 7 pounds, and no motion results. Dynamics treats of the action of *unbalanced* force systems on bodies. When the force systems are unbalanced, the body will always move as a result of the force action. To lift the coat off the hook requires a force slightly greater than 7 pounds—the weight of the coat.

The bodies of matter on which the force actions take place may be in the *solid state,* the *liquid state* or the *gaseous state.* Matter in the solid state is distinguished by the possession of a definite shape and of a definite volume, in the liquid state it is distinguished by having a definite volume but not a definite shape, while in the gaseous state it has neither a definite volume nor a definite shape, ordinarily expanding until it fills any container in which it may be placed. Gases and liquids are grouped together as fluids, since, being without internal rigidity, they flow readily.

In our study of mechanics we will first take up the mechanics of liquids. This subject contains two subdivisions, Hydro dynamics—the study of force actions in liquids in motion, and Hydrostatics—the study of force actions in liquids at rest.

REVIEW.

1. What is meant by the statement that the density of a body is the mass per unit volume?
2. Explain the difference between force and energy.
3. What is the doctrine of the conservation of matter and energy?
4. What are the three states in which matter is found?

CHAPTER I

THE MECHANICS OF LIQUIDS

1. Hydrostatics.— The study of force actions in *moving* liquids is a subject of peculiar interest to the engineer who may wish to design water wheels or similar machinery. In our study of the mechanics of liquids we shall confine our attention to hydrostatics, that is to cases in which bodies of liquid are at rest. Here, as elsewhere, in dealing with similar phenomena, we must distinguish carefully between the total force acting on an area—as, for example, on the bottom of a pail containing water—and the pressure on that area. We shall define pressure as the force per unit area. The total force will be expressed in grams or in pounds—the pressure will be stated in grams per square centimeter or in pounds per square inch.

Let us suppose that we have a completely closed vessel full of water. Let there be in one side of this vessel a movable piston *B* Fig. 1. If we apply a force to the piston, this force will be transmitted thruout the liquid and will act on the walls of the containing vessel. An experimental study of the resulting pressures would lead us to the following conclusion: Neglecting effects due to the weight of the liquid, the pressure, that is the force per unit area, is the same at all points on all the containing walls and is the same in magnitude as the pressure applied to the movable piston. The total force action in grams or pounds on any specified area of the containing walls is of course equal to the product of the

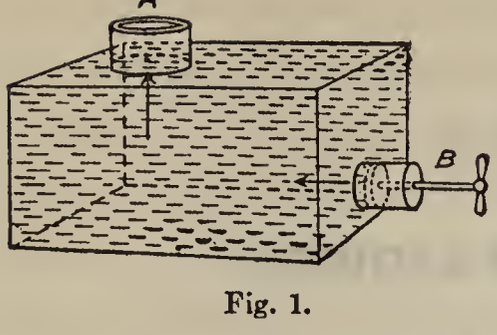

Fig. 1.

pressure p (in grams per square centimeter or in pounds per square inch) into the area A (in square centimeters or square inches) of the specified area. If the specified area be made movable, it will move outward and will exert a force equal to the product pA which will be greater than the force applied to the piston B if the area of the movable piece is greater than the area of the piston. This discovery was first made and stated by Blaise Pascal in 1653.

2. Pascal's Law.— The facts brought out in the last paragraph can be extended by further experiment to give the following statements:

(a) Pressure applied to an inclosed liquid is transmitted undiminished in all directions thruout the liquid.

(b) The pressure at any point in the body of the liquid acts equally in all directions.

(c) The pressure on each containing wall acts at right angles to the wall.

Statement (a) is a restatement of the principle ascribed above to Pascal. Statement (b) is necessarily true because if the pressure at the selected point were not equal in all directions then there would be an unbalanced force in some direction which, since liquids do not possess any rigidity, would cause the liquid to move and this is contrary to the observed condition of affairs. Statement (c) is also of necessity true, because if the resultant pressure on the boundary walls were not at right angles to the walls, there would be a component of the pressure along the wall which would cause the liquid to move in the direction of the component, a movement which does not take place in fact.

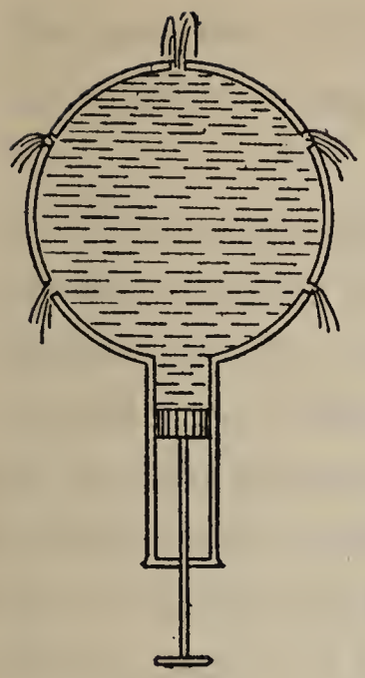

Fig. 2.

When pressure is applied to the piston, the water squirts out of all the holes, showing that the pressure is transmitted in all directions.

The hydraulic press is a machine in which Pascal's Principle is applied to obtain the enormous pressures required in baling hay or cotton, in forcing car wheels on to axles and in doing similar work. This press consists of two cylinders joined with a pipe. One of the cylinders *A* is of large area. It contains a piston *C* arranged to act on the material which is to be compressed in the way shown in Fig. 3. The other cylinder *B* contains a piston of small area *D* which can be lifted and depressed by means of a lever. When the small piston is depressed, water is forced thru the pipe into the large cylinder. When the small piston is raised, check valves prevent the water in the large cylinder from flowing back. During the down stroke the two cylinders are in free communication thru the pipe. If the area of the small piston is $\frac{1}{2}$ sq. in. and the area of the large piston is 2 sq. ft., that is 288 sq. in., then a force of 10 lb. applied to the small piston will produce a pressure of 20 lb. per sq. in., in which, acting on the 288 sq. in. of area of the large piston, will develop a force on that piston of 5760 lb.

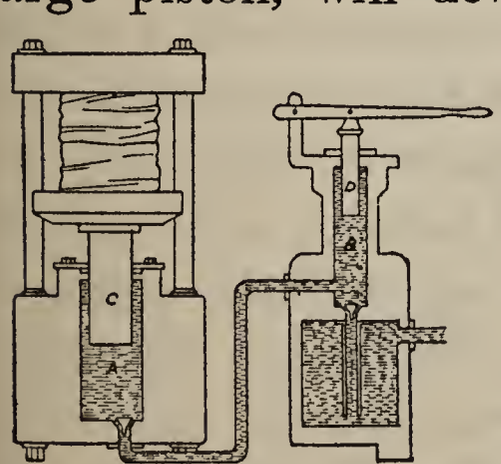

Fig. 3.

Here as elsewhere in this Text we shall assume water to be entirely incompressible—that is to say we shall assume that its volume remains unchanged whatever pressure may be applied to it. Since a force of 21.5 tons

applied to a cubic inch of water reduces the volume only about one-tenth, this assumption is from our point of view quite justifiable.

It will be noted that the rise of the large piston in the press is due to the extra water forced into the large cylinder due to the downward motion of the small piston. The increase in water volume in the large cylinder is therefore equal to the decrease in water volume in the small cylinder and the downward travel of the small piston must be greater than the upward travel of the large piston in the same proportion that the area of the larger piston is greater than the area of the smaller. These presses are therefore very slow in action and can be used for work only where speed is not an essential. In the press described above, the small piston must travel a distance of 576 feet in order to raise the large piston just one foot.

3. Depth and Pressure.—In the preceding discussions of the transmission of pressure in liquids, we have purposely neglected any consideration of the force effects due to the weight of the liquid itself. It is evident, however, that very considerable pressures must exist in the lower layers of any liquid due merely to the weight of the upper layers. The existence of these pressures can be demonstrated by thrusting a tin cylinder with a closed bottom in which there is a nail hole, downward into a pail full of water. Water will spurt up thru the hole. The height of this water jet will increase as the depth of immersion is increased, showing that the pressure becomes greater with the depth. We can readily calculate the total force due to this cause acting on the bottom of a cylindrical vessel full of liquid. This total force is exactly the weight of the liquid. To determine this

weight, we multiply the area of the bottom of the vessel A, by the height of the liquid h, thus obtaining the volume of the liquid. This volume multiplied by the weight of a unit volume of the liquid gives the total weight on the bottom and therefore the total force. The weight of a unit volume is called the density of the liquid d. The formula expression for the total force F is therefore $F = Ahd$. This expression is usually called the Law of Gravity Pressure. If A is in cm.^{2}, h in cm. and d in $\frac{\text{grams}}{\text{cm.}^3}$, then F will be in grams. The pressure p on the bottom, that is the force per unit area, will equal $\frac{F}{A}$ and we can therefore write $p = hd$.

It is evident that the pressure over any horizontal plane in a liquid at rest must be the same at all points in the plane. If this were not so, the liquid would flow and would no longer be at rest. It can be seen furthermore that the *free surface of a liquid must be horizontal,* for if it were not, then the pressures at two points in the same horizontal plane would have different values, due to the greater depth of the liquid at one of the points, and the liquid would flow until the surface became truly horizontal.

The two formulæ $F = Ahd$ and $p = hd$ enable us to calculate the total force and the pressure on any horizontal surface at any depth in a liquid of any density. We must now consider how to calculate the force action on the vertical rectangular side of a vessel containing a liquid. The pressure at the surface of the liquid is zero because h is zero—the pressure at the bottom of the side is hd where h is the total depth. In going down the side from the top surface to the bottom, the pressure increases uniformly with the depth, being at any depth, h, equal to hd. The average value of the pressure will

therefore be the pressure half-way down at the depth $\frac{h}{2}$, and will equal $\frac{h}{2}d$. If we multiply the total area of the side by this average value of the pressure, we shall have a valid expression for the total force action on the vertical side of a tank full of water, $F = \frac{Ahd}{2}$.

This same formula can be used in a slightly modified form to determine the force action on any regularly shaped portion of a vertical containing wall, merely multiplying the area of the portion of the wall by the density of the liquid and by the average depth, that is by the depth down to the center of area of the portion on which the force is to be calculated. For example let it be required to calculate the total force action on a square sluice gate 4 feet on an edge in the face of a dam—the upper edge of the gate being parallel to the water surface and 15 feet below that surface. Here $A = 16$ ft.^{2}, $d = \frac{62.4 \text{ lb.}}{\text{ft.}^3}$ and the depth to the center of area equals 17 ft. The total force action on the gate is therefore 16972.8 pounds.

It can be stated from the facts given above that the pressure at any point in a vessel of any shape whatever or in any system of communicating vessels or pipes containing a liquid at rest *is equal to the density of the liquid multiplied by the depth of the point below the free surface of the liquid.*

This principle enables us to see that liquid poured into a set of communicating pipes, like that shown in Fig. 4, will rise to the same level in all the branches,—the pressure at the bottom of any branch as at A will equal hd where h is the height above A of the liquid in the branch where it was introduced, and this pressure will obviously just suffice to support in the pipe above

Fig. 4.

A a column of liquid of the same height *h*. This fact of common observation is covered by the well-known statement that "water seeks its own level." For example if a city water system is supplied from a reservoir on a hill, the water will rise in the buildings to a height equal to that of the water surface in the reservoir but no higher. Of course, when the water is flowing thru great lengths of small pipes, the effective pressure, or effective "head of water," is considerably decreased by friction in the pipes.

The flow of water from artesian wells is maintained in the same way. The head of water in these wells results from the pressure due to water flowing down inclined strata from hills or mountains which may be many miles distant.

In our study of the pressure effect in liquids due to the weight of the latter we have not considered the simultaneous action of external forces on the body of liquid. It is evident that in order to determine the total pressure at any point in a liquid we must add to the pressure hd due to the weight of the liquid above, whatever pressure, P, may be acting on the free surface because this pressure P, will be transmitted undiminished thruout the body of the liquid. If, for example, we wish to find the total pressure at the depth of a mile in the ocean, we must first set down the total pressure due to the weight of the atmosphere on the free surface of the water. This pressure is about 15 lb. per in.^{2} or 2160 lb. per ft.^{2} To this we must add the pressure hd due to the weight of the water. The density of sea water is 63.9 pounds per cubic foot, whence the pressure per square foot, at

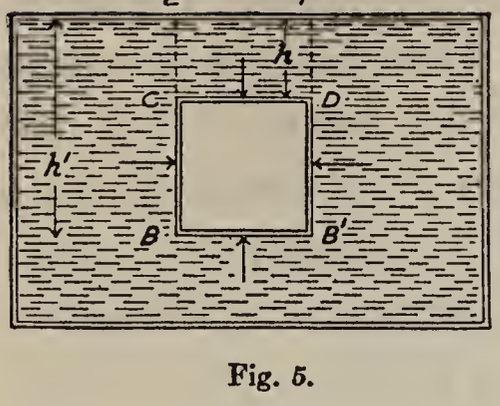

Fig. 5.

the depth of a mile, due to this cause is 5,280 (the number of feet in a mile) multiplied by 63.9 or 337,392 pounds per square foot. The total pressure at the depth of a mile is therefore 339,552 pounds or about 169.7 tons per square foot.

4. Archimedes' Principle.—We shall next examine the force actions on a solid body immersed in a fluid at rest. Let us suppose that we have a cube of any material held with one of its faces horizontal below the surface of water as shown in Fig. 5. The pressure on the sides of the cube will be equal and opposite on the opposite pairs of faces, so that the cube will experience no unbalanced lateral forces. The upward pressure on the lower face BB' of the cube will be $h_1 d$. If the area of the face be A, the total upward force acting on the bottom of the cube will be $Ah_1 d$. The pressure acting down on the top will be hd and the total downward force will be Ahd. It is evident that $Ah_1 d$ is greater than Ahd so that there will be an unbalanced force equal to $Ah_1 d$ minus Ahd acting upward on the block. If this unbalanced force is greater than the weight of the block, the block will rise if released—if the force is less than the weight of the block, the block will sink if released, or if the unbalanced force is exactly equal to the weight of the block, the block will remain stationary if released. Now we can easily see that the term $Ah_1 d$ equals the weight of a column of water with an area A and a side $BB'FE$ and that the term Ahd equals the weight of a column of water of area A and side $CDFE$. The difference of these two terms is clearly equal to the weight of a col-

umn of water of area A and side $BB'CD$—that is, to the weight of the water displaced by the cube itself. This same demonstration can be made, tho not so easily, for a body of any shape whatever so that we can say that *any body immersed in a liquid is buoyed up by a force equal to the weight of the liquid displaced by it.*

This law was first formulated by Archimedes and is always known as the Principle of Archimedes. Archimedes was a Greek philosopher (born 287—died 212 B. C.) who lived in Syracuse, Sicily.

5. Determinations of Density.— The density of a body has already been defined as the quantity of matter in a unit volume or as the *mass per unit volume.* Since at any one place on the earth's surface the weights of bodies are proportional to their masses, the ordinary way of determining the mass of a body is to weigh it. The mass is then expressed in grams or pounds. To calculate the density, it is necessary in addition to know the volume of the body. If we immerse the body in water and determine its loss of weight in grams, we have, directly by Archimedes' law, the weight in grams of an equal volume of water. Now in the metric system a gram is equal to the weight of a cubic centimeter of pure water at 4° Centigrade, so that the loss of weight in grams is seen to be numerically equal to the volume of the body in cubic centimeters. We can thus always determine the density of a body *by dividing its weight in air by its loss of weight in water.* If this work is carried out in metric units, the quotient will be the density in grams per cubic centimeter. If the work is carried out in pounds, the loss of weight must first be divided by the weight in pounds of a cubic foot of water (62.4 lb. per ft.^{3}) before applying the foregoing rule.

The density will then come out in pounds per cubic foot.

6. Specific Gravity.—The specific gravity of a body is defined as the *ratio of the weight of the body to the weight of an equal volume of water.* This is a mere number and—like all ratios—has no units. It will therefore be the same number in whatever system of units the measurements are made. Since, as was just stated, the weight of a quantity of water in grams is numerically equal to its volume in cubic centimeters, it is evident that the specific gravity of a body is numerically equal to its density when the density is expressed in the metric system. To get the proper value of the density in the English system, the specific gravity must be multiplied by 62.4. For example any object made of a certain grade of cast iron weighs seven times as much as an equal volume of water. The specific gravity of this kind of cast iron is therefore 7—its density in the metric system is 7 grams per cubic centimeter and its density in the English system is 436.8 pounds per cubic foot.

In all problems of design and in all calculations dealing with the weights or volumes of material, a knowledge of the density or specific gravity of the materials is of first importance. It is therefore worth while not only to know definitely what these terms mean but also to be familiar with some of the methods by which the quantities are determined. Let us now consider briefly the ordinary experimental method used for determining the densities (a) of solids that sink in water, (b) of solids that float in water.

7. Densities of Solids and Liquids.—(a) If the solid will sink in water it is only necessary to weigh it in air—

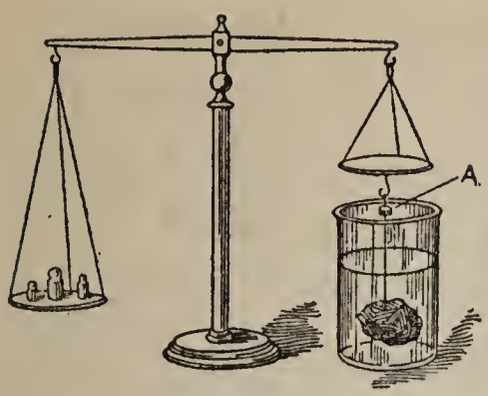

Fig. 6.

then to immerse it in water and determine its loss of weight. As already stated, the specific gravity $= \frac{\textit{the weight in air}}{\textit{loss of weight in water}}$. An ordinary equal arm balance can be used in this work. If the instrument is specially adapted to the work by raising one of the pans as in Fig. 6, it is frequently called a hydrostatic balance.

(b) Fig. 6 shows the hydrostatic balance in use for determining the density of a solid at A which will not sink in water. First the solid is weighed in air. Call this weight W. Then a sinker heavy enough to sink the body is attached, and the weight is determined when the sinker is in the water and the body is in the air as shown in the figure. Call this weight W_1 grams. Then immerse both solids and determine the weight. Call this weight W_2 grams. The difference $W_1 - W_2$ is evidently caused by the buoyant effect of the water on the light solid, and by Archimedes' Principle this is equal to the weight of the water displaced by it in grams and is numerically equal to the volume of the light body in cubic centimeters. The specific gravity or the density in the metric system is seen to be given by the expression $\frac{W}{W_1 - W_2}$.

The following table of densities will give some idea of the range of values for ordinary materials:

	Specific Gravity	*Weight lb./ft.*^{3}
Water	1.	62.4
Seawater	1.026	63.9
Aluminum, cast	2.56	159.8
Iron, cast	7.23	451.0
Tin, cast	7.29	455.

	Specific Gravity	*Weight lb./ft.*^{3}
Brass, cast	8.4	524.4
Copper, cast	8.6	537.3
Lead	11.4	711.36
Mercury	13.6	848.75
Gold	17.7	1106.42
Platinum	21.2	1322.8
Ash	.69	43.
Mahogany, Spanish	.85	53.
Glass	2.7	170.
Marble	2.7	170.

8. Law of Flotation.—When a block of wood is placed on the surface of a body of water it begins to sink. After a certain portion of it is submerged, it will float and remain stationary. The fact that it remains stationary shows that the force acting upward, equal to the weight of the displaced water, must exactly equal the weight of the block. In other words the *block sinks* until it *displaces a volume of water equal in weight to its own weight.* This statement is known as the *Law of Flotation.* It is of course merely a special form of Archimedes' Principle. It will be noted that the earth's pull downward on the floating body is exactly neutralized by the push upward of the displaced water. A floating body is therefore without any effective weight.

The principle of the law of Flotation is made use of in determining the densities of liquids by means of the *hydrometer.* The hydrometer consists of a glass bulb weighted so that it will float upright. The bulb has a long stem with graduations on it. (See Fig. 7.) When the instrument is placed in a liquid, it will sink until it displaces its own weight of the liquid. The weight of

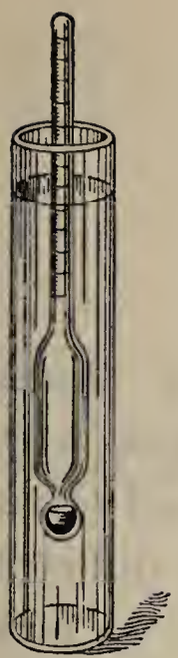

Fig. 7.

the hydrometer is of course constant, consequently it will sink to different depths in liquids of different densities. The less dense the liquid, the deeper the instrument must sink in it before it will displace a volume of the liquid that weighs as much as the instrument itself. In practice these instrumentsare graduated by placing them in test solutions the densities of which have been determined by other methods. The depths to which they sink are then marked on the stem and serve as reference points. The specific gravity of gasoline and of the liquid in automobile storage batteries is ordinarily determined with instruments of this type. The readings on the stems of these instruments usually give the specific gravities directly, but they are sometimes graduated according to an arbitrary scale to adapt them to some special use, consequently a new one must always be examined critically before being put into service.

Before leaving this section of our subject it will be well to point out that the laws here developed for liquids, namely Pascal's Law—the Law of Gravity Pressure—and Archimedes' Principle, are all generally applicable to *fluids;* that is to gases as well as to liquids. Gases, however, are so easily compressible that they do not act in general like liquids, and it will be necessary therefore to consider their special properties at greater length in another chapter.

REVIEW.

1. State and explain Pascal's Law.
2. Describe the hydraulic press.
3. How do you calculate the pressure at any point in a vessel containing a liquid at rest?
4. Define and distinguish between specific gravity and density.

CHAPTER II

THE MECHANICS OF GASES

1. Atmospheric Pressure.—In our study of force actions in gases we may consider air—the gas with which we are all most familiar—as typical of all other gases. The density of gases is so slight that it is difficult to realize that they have an appreciable weight. Yet a cubic foot of air weighs more than an ounce and twelve cubic feet weigh very nearly one pound. Consequently the amount of air in a room 60 ft. by 30 ft. by 15 ft. weighs more than one ton. At the temperature of freezing and under normal pressure the exact density of air is .001293 grams per cubic centimeter or 1.293 grams per liter. The liter, it will be remembered, is 1000 cubic centimeters, a trifle more than one quart. The specific gravity of air is $\frac{1}{773}$—that is to say: 773 cubic feet of air weigh as much as 1 cubic foot of water.

Since in point of fact we live at the bottom of a sea of air many miles deep, it is evident upon consideration of the facts just stated taken together with the principles of fluid pressures which we studied in Chapter I, that a large pressure caused by the weight of the atmosphere must exist all about us on the surface of the earth. The existence of this pressure can readily be demonstrated by stretching a rubber membrane over the open end of a glass jar as shown in Fig. 8 and then sucking out the air from the inside of the jar, either with the mouth or with a special pump. The weight of the outside air will immediately depress the membrane very noticeably.

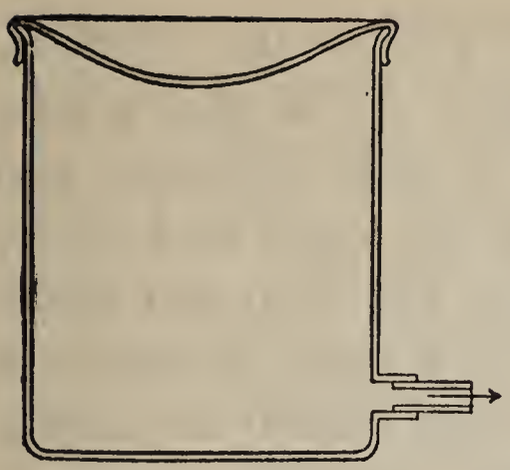

Fig. 8.

This pressure due to the weight of the air is also demonstrated in sucking lemonade thru a straw. You draw some of the air out of the top of the straw by enlarging the mouth cavity, and the pressure of the atmosphere on the liquid surface then pushes the lemonade up into the mouth. Mechanical applications of this method of raising liquids were in wide use as long ago as the time of Aristotle (fourth century B. C.) long before the real cause of the rising was understood. The ancients explained the phenomenon by the somewhat vague statement that "nature abhors a vacuum"—a space exhausted of air being known as a *vacuum*. The famous philosopher Galileo (1564—1642) probably grasped the true reason for the rise of liquids in exhausted tubes altho he never specifically stated it. About 1640, the Duke of Tuscany, in attempting to pump water out of a deep well which he had dug near Florence, found that no pumps could be made to draw the water higher than about 32 feet above the water level in the well. When this was reported to Galileo, the philosopher remarked that "evidently nature's horror of a vacuum did not extend beyond 32 feet." He devised an experiment to test this matter, but died before the experiment was completed.

The test was actually performed by his pupil, Torricelli, in 1643. Torricelli reasoned that if the pressure of the atmospheric air was responsible for forcing liquids up into exhausted tubes, then, if water was pushed up 32 feet, mercury, which is about thirteen times as heavy

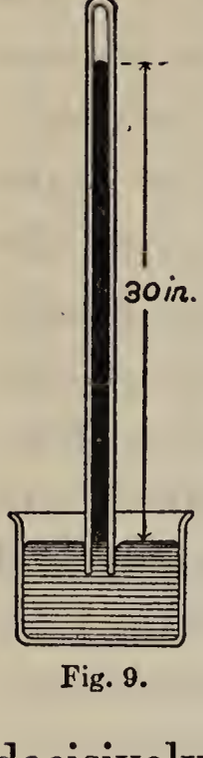

Fig. 9.

as water, ought to be pushed up about $\frac{1}{13}$ as far or about 30 inches. So he took a tube 40 inches long, filled it with mercury and holding his thumb over the open end of the tube, inverted it and put the open end below the surface of mercury in a vessel as shown in Fig. 9. On removing his thumb he found, just as he had expected, that the mercury level fell to about 30 inches above the surface in the dish. That the mercury is actually held up in the tube by the pressure of the external air and by no other means can be decisively demonstrated by covering the dish in Torricelli's apparatus with a bell jar as shown in Fig. 10 and then sucking the air out of the bell jar. The level of the mercury column will be found to fall gradually as the air is drawn out and the pressure thus reduced.

2. The Barometer.— Pascal, the French philosopher, mentioned in Chapter I, tested this matter directly by carrying a mercury device, which we will call a barometer or pressure measurer, similar to Torricelli's, to the top of a high tower in Paris. He found that the level of the mercury column fell somewhat. In following up his experiment he requested his brother-in-law, Perrier, who lived in the south of France, to carry a barometer to the top of the Puy de Dome, a high mountain. Perrier reported in a letter which is still in existence that he "was ravished with admiration and astonishment" on finding that for an ascent of 1000 meters, the mercury sank in

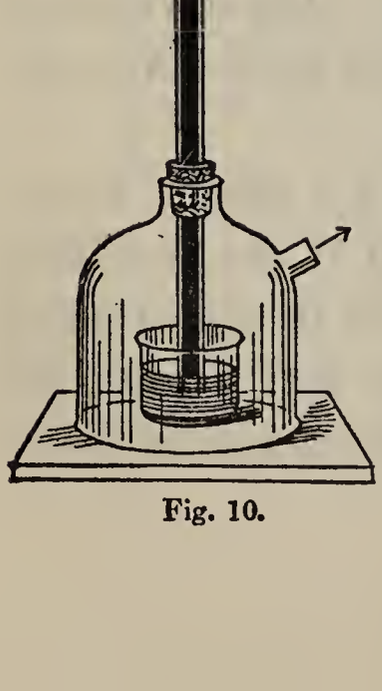
Fig. 10.

P = pressure of air thru open end of tube.

Fig. 11.

the tube about 8 centimeters. More accurate figures, subsequently determined, show that the change in the mercury level is one centimeter for every 120 meters of ascent.

The pressure of the atmosphere as determined with a barometer at any place fluctuates from hour to hour. At sea-level the range of heights lies between 28 and 32 inches. The average height taken over a long period at sea level in 45° north latitude has been found to be 76 centimeters. This is called the "normal height of the barometer" and the corresponding air pressure is called "normal atmospheric pressure." The value of this pressure in grams per square centimeter can readily be calculated from the formula $p = hd$, since the downward pressure of the mercury volume must exactly equal the upward pressure transmitted to its bottom from the free surface on which the air is pressing. The density of mercury is 13.596 grams per cm.^{3} The pressure at the bottom of a mercury column 76 cm. high is therefore $76 \times 13.596 = 1033.3$ grams per cm.^{2} This is equal to 14.7 lb. per square inch.

There are two types of barometers in general use, the mercury barometer (Fig. 11), which is in all essential features the same as Torricelli's original apparatus, and the aneroid barometer (Fig. 12). The aneroid barometer contains a partially exhausted flat box *D*, the lid of which acts upon a pointer *B* moving over a dial, by means of a

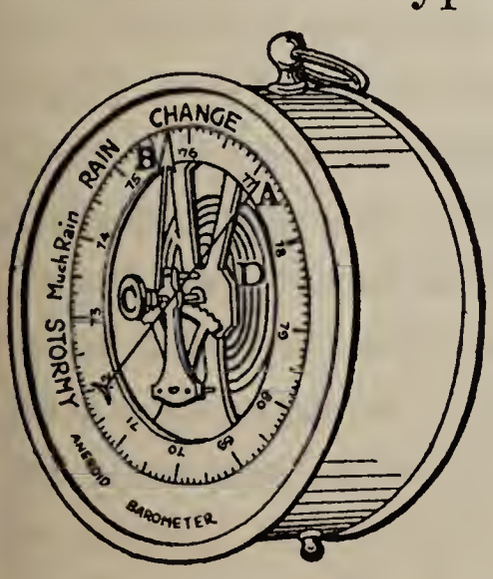

Fig. 12.

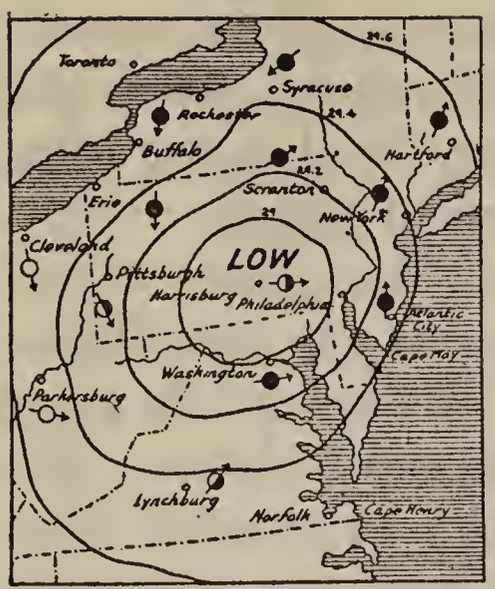

Fig. 13.

series of wheels and levers which very much magnify the slight movements of the lid in and out under the variations of the air pressure. These instruments can be made as small as a watch and may be carried in the pocket. They are very useful in enabling mountain climbers or aviators to know the heights which they have reached. If the pointer is provided with a pen or pencil which traces a mark on a paper carried by a drum rotated by clockwork, we have the "recording barograph," which gives a permanent record of the changes in air pressures over any period. These devices when used in air planes are called "altimeters." They are used to test the performance of the planes as well as of the pilots.

3. Weather Predictions.—The connection existing between changes in the barometer height and changes in the weather is of interest and importance to everybody. Weather predictions are based very largely on measurements of the atmospheric pressure taken simultaneously over large areas, as for instance at all the stations of the U. S. Weather Bureau thruout the country. These readings are telegraphed to headquarters at the same hour each day, and when the values are plotted on a map and continuous lines (called isobars) are drawn connecting all the points where the pressure is the same —one line for each pressure—a figure results something like that shown in Fig. 13. Stormy areas generally go with areas of low pressure which lie at the bottom of more or less circular depressions in the atmosphere.

The winds blow around these circular depressions so as to form " whirlpools " or " cyclones."

These cyclonic centers of low pressure move along pathways fairly well known as the result of long observation and known to lie generally from southwest to northeast. If a cyclonic area is observed in the southwestern part of the United States and its path and velocity are determined or accurately guessed by the weather bureau officials, then the time of its arrival in other parts of the country can be foretold. The individual observer who has a household barometer can detect the approach of one of these storm centers by observing the fall of his barometer. Rises in pressure are on the other hand indicative of coming fair weather. Predictions made in this way naturally apply only to large general storms and not to local weather changes—small showers and so on.

4. Internal and External Pressure.—It is known to all that air is very compressible and that when compressed it will exert a great pressure. We know, for instance, that a large amount of air can be pumped into an automobile tire and that as a result of this the tire becomes very hard so that it is difficult to make a dent in it. If the pressure is released by depressing the valve stem, the air will rush out until the inside air pressure equals the air pressure outside. It will be noted that when the outward pressure of the air inside a hollow container is the same as the inward pressure of the air outside, there is no resultant force acting on the wall of the container, hence there is no tendency to rupture or displace the wall. The ordinary atmospheric pressure of 14.7 lb. per square inch is very nearly equal to one ton per square foot. If we take the area of the human body to be about

10 square feet, which is below the average, then it is seen that a total force of about 10 tons is acting continuously on the exterior of every one of us. The reason we don't experience any discomfort from this pressure is that all the internal cavities and fluids of the body contain air which presses out with an equal pressure.

5. Boyle's Law of Pressure and Density.— The first accurate investigations of the relation existing between the volume of a mass of inclosed gas and the pressure applied to it were made in 1662 by Robert Boyle (1627-1691) an Irish physicist. The same relation was worked out by the Frenchman, Mariotte, fourteen years later and is known in France as "Mariotte's Law." In this country, however, we refer to it always as "Boyle's Law."

The facts discovered by Boyle can be stated as follows:

The temperature remaining fixed, the volume of an inclosed mass of gas varies inversely with the pressure applied to it.

It will be noted that in accordance with this law doubling the pressure reduces the volume of the inclosed gas to one-half. Since the total quantity of matter in the inclosed gas is the same under all conditions, the quantity of matter per unit volume (or density) must become twice as great when the volume is reduced to one-half. It is therefore plain that doubling the pressure will double the density and that tripling the pressure will triple the density and so on—an example of a direct variation.

In consequence of these facts Boyle's Law is frequently stated in the following form: *The temperature remaining fixed, the density of an inclosed mass of gas*

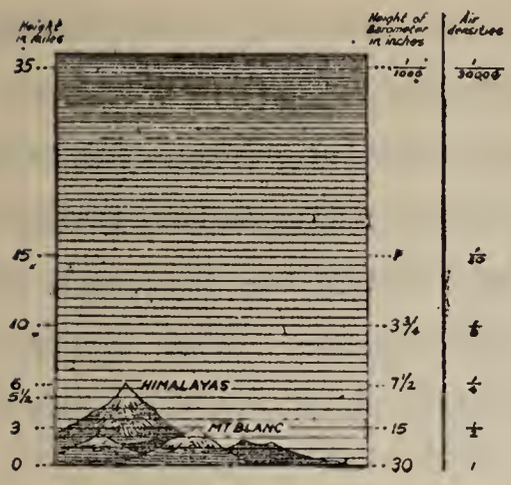

Fig. 14

is directly proportional to the pressure acting on it.

We are now in a position to understand the difference between the law connecting depth of immersion and pressure in liquids, and the law connecting depth and pressure in air or any other gas. At any point below the surface of a body of gas there will be, just as in liquids, a pressure due to the weight of the gas above; this pressure will be equal in all directions and will act at right angles to the walls of any containing vessels but it will not be directly proportional to the depth of immersion as is the case in liquids. This is because the density of gases increases with the pressure so that the pressure effect multiplies itself progressively as we descend to greater depths in a gas. The exact law connecting the gravity pressure in a body of gas with the depth of immersion in the gas is complicated and need not be stated here, but the general character of the relation can be studied out by reference to Fig. 14.

6. Height of the Atmosphere.— It is interesting to note that if the density of the air remained the same at all heights and was equal to the density observed at sea level, then the total height of the atmosphere as calculated from the barometric height at sea level would be just about five miles. The peaks of the Himalayas would rise above it. This height to which the air would extend if it were incompressible like water is called *the height of the homogeneous atmosphere.*

Many explorations of the upper atmosphere have been made by aeronauts in balloons and air-planes and

also by the use of small test balloons carrying recording instruments provided with parachutes. These test balloons which are only two or three feet in diameter are partly filled with hydrogen and rise for the same reason that a cork rises in water, until they burst on account of the expansion of the inclosed hydrogen. The instruments descend safely to the ground in a small basket borne by a parachute.

The greatest height ever reached by men in a balloon is a little more than seven miles. At that elevation the height of the barometer is only 7 inches and the temperature is 60° Fahrenheit below zero. The greatest height ever reached by a test balloon is 18.95 miles. This height was reached in September, 1910, by a balloon sent up from the U. S. government observatory at Mount Weather, Virginia.

Fig. 14 gives definite figures collected by the methods described in the foregoing. It shows the air pressures and densities at different heights. It will be noted that at a height of 35 miles the air density is only $\frac{1}{30000}$ of that at sea level, so that at such heights there is practically no air. Nevertheless, it will be understood that there is no definite upper limit to the atmosphere—the air merely getting thinner and thinner as the distance from the earth becomes greater. The final limit beyond which there is absolutely no air is estimated at anywhere between 100 and 500 miles. These estimates are based largely on the height at which meteors or "shooting stars" first become visible. These meteors are small solid masses which, moving with enormous speed, enter the earth's atmosphere and are heated to incandescence by the friction of the air.

If a boring miles deep could be made in the earth, the

density of the air as we descended the hole would increase just as rapidly as we have seen it decrease as we ascend above the earth's surface. At a depth of 35 miles it would consequently be 1000 times as dense as at the surface of the earth so that wood and water would float in it.

7. The Balloon.—The physical principles which we have considered in the previous discussion enable us now to study the construction and mode of operation of a number of appliances of great practical importance.

We have already spoken of the ascents of balloons to great heights. A balloon is merely a gas-tight bag of varnished silk or of rubberized fabric which is inflated with some gas less dense than air. When inflated, it is buoyed up in accordance with Archimedes' Principle, by a force equal to the weight of the air displaced. If this force is greater than the weight of the balloon-casing, net and basket plus the weight of the gas in the balloon, the balloon will tend to rise. Since air weighs 1.29 kg. per $m.^3$ and hydrogen, the lightest known gas, weighs only .09 kg. per $m.^3$, a total buoyant force of 1.20 kg. per $m.^3$ of displacement can be obtained by using hydrogen. Illuminating gas which ordinarily weighs about .75 kg. per $m.^3$ gives a buoyance of only about .54 kg. per $m.^3$, but since it is very much cheaper than hydrogen, it is used in nearly all ordinary cases.

Military balloons of which large numbers were used in the recent war were however filled with hydrogen to make them ascend more rapidly. German Zeppelins which were large cigar-shaped balloons provided with a rigid metallic frame-work containing a number, usually 15 to 20, separate small gas bags inside the sheathing, were also filled with hydrogen. Since this gas is very

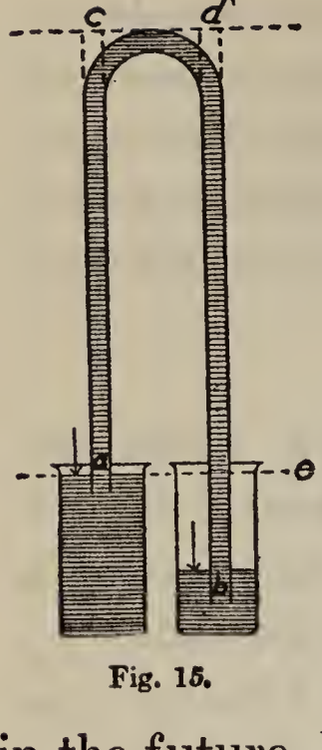

Fig. 15.

inflammable, it is for that reason not well suited to military use, but it was the best that could be had at the time. In the last year of the war, helium gas was prepared for the first time in large bulk by extracting it from natural gas in Texas. Helium is entirely non-inflammable and, having a density of 0.18 kg. per m.^{3} gives a buoyant effect of 1.06 kg. per cubic meter of displacement, which makes it comparable with that of hydrogen. It is probable that in the future, balloons—at any rate those used for military purposes—will be inflated with helium.

8. The Siphon.— The siphon is a bent tube open at both ends with one arm longer than the other, filled with liquid and placed in an inverted position with the shorter arm in a vessel containing liquid as shown in Fig. 15. The liquid will flow out of the longer arm until the level of liquid in the vessel falls to the end of the shorter arm. This device is very useful for transferring liquids from one vessel to another without disturbing the vessels. Either the lower or the upper layers of liquid can be drawn out as desired. The cause which maintains this flow can be understood by computing the pressures at *a* and *b*. At *a* the pressure equals the atmospheric pressure transmitted from the free surface outside minus the weight per cm.^{2} of the column of liquid *ac*. At *b* the pressure equals the atmospheric pressure minus the weight per cm.^{2} of the column of water *db*. It is evident that the net pressure at *b* is less than at *a* by an amount equal to the weight of the column of liquid *eb*, consequently the liquid is pushed around over the bend pro-

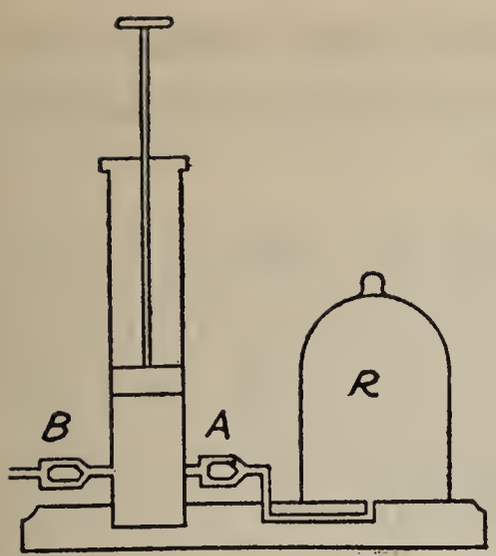

Fig. 16.

vided the bend is not at a height greater than the barometric height for the particular liquid above the surface in the vessel. Water for example cannot be siphoned over a bend higher than about 32 feet. It is evident that the flow will cease when the level in the two vessels becomes the same because then the weight of the column of liquid *eb,* which constitutes the driving force, will be 0. The tube must be completely filled with liquid before it is put in position. A study of the explanation given will show that it is not essential that the arms of the pipe be unequal in length: it is necessary only that the level of liquid around the delivery arm be lower than the level around the intake arm.

9. The Air Pump.—Fig. 16 shows a simple air pump for pumping the air out of the receiving vessel *R.* When the piston is raised, the air in the receiver expands thru the valve *A* and fills the cylinder, the external air being prevented from entering by the valve *B.* The valves are conical plugs accurately ground into circular seats. The air passes thru freely in one direction but any flow in the other direction is cut off by the movement of the plug into its seat. When the piston is depressed, valve *A* closes and the air in the cylinder is pushed out thru *B.* This operation is repeated as often as desired, a certain fraction of the air remaining in the receiver being removed on every up stroke. It will be observed that it is impossible to remove all the air by the use of this device. In order to get the very high vacuum, that is to say, the very low pressure, used in

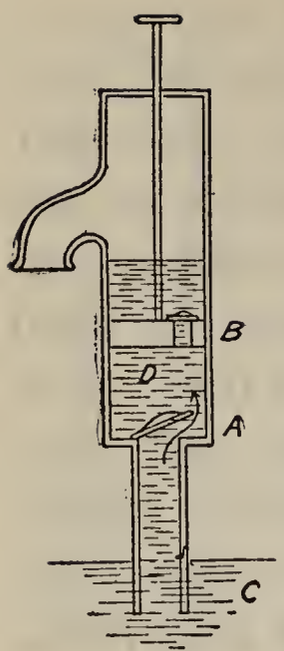

Fig. 17.

such things as electric light bulbs, very special and complicated forms of pumps must be used.

10. The Lift Pump for Water.—This pump, Fig. 17, which is the common type of water pump used over wells and cisterns, is essentially similar to the air pump in construction. When the piston is lifted, the air pressure in the cylinder *D* is reduced and the atmospheric pressure on the water in the cistern *C* pushes the water up into the cylinder. When the piston descends, valve *A* prevents the water from running back into the cistern and valve *B* opens and allows the water to flow thru to the top of the piston. On the next up-stroke valve *B* closes and the water is lifted to the level of the spout out of which it runs. Of course no pump of this type can lift water from a well in which the surface of the water is more than about 32 feet below the cylinder, for 32 feet is the greatest height to which the pressure of the air can push the water.

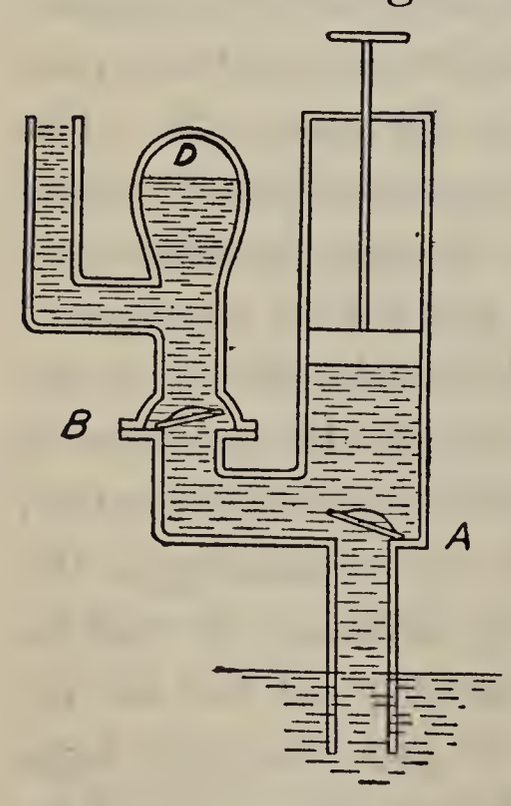

Fig. 18.

11. The Force Pump for Water.—If the well is deeper than 32 feet, the pump is modified as shown in Fig. 18. The cylinder itself is put down in the well within ten or fifteen feet of the water surface and the piston is driven by a long rod. The action of the pump is easily understood from the figure. On the down stroke the water is forced out thru valve *B*, valve *A*

being closed, and up the pipe to a height limited only by the power available for pushing the piston and by the strength of the cylinder and pipes. In order to make the water delivery of a pump of this type steady instead of pulsating with every stroke, an " air dome " D is introduced as shown in the figure. The air compressed in the dome during the down stroke of the piston expands during the up stroke and thus maintains the water flow.

12. The Diving Bell.—The diving bell is a rigid bell-shaped casing which sinks in water by its own weight. It is used to protect workmen in underwater constructions as of bridges, piers, etc. Air is pumped into the bell thru a hose entering the top. A constant escape of air over the edge of the bell is maintained. The pressure of the air in the bell must of course equal the pressure of the water outside. To work at a depth of 100 feet, an air pressure of three atmospheres will therefore be necessary. These high pressures seriously affect the comfort and health of the workmen. In practice, work is seldom carried out at depths greater than 60 feet, altho 80 feet is regarded as safe. In working on the bridge over the Mississippi at St. Louis, however, the bells were sent down 110 feet without mishaps. The *diving suit* is made of rubber with a metal helmet and heavily weighted shoes. In the most modern outfits the diver carries his air supply, under a pressure of 40 atmospheres, in a tank on his back. This air is admitted into his suit as needed. In order to keep the water out of the suit, the air pressure inside must roughly equal the water pressure outside. When the diver wishes to rise, he admits enough air to the suit to make himself float. Cases are on record where divers have descended to depths of slightly over 200 feet.

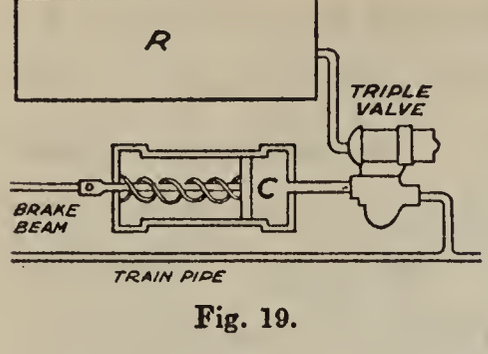

Fig. 19.

13. The Air Brake.—The air brake has had more to do with the development of high speed transportation than any other single device. The essential parts of the Westinghouse air brake system are shown in Fig. 19. By means of a pump on the locomotive an air pressure of about 90 pounds to the square inch is maintained in the train pipe, which passes along from one end of the train to the other. As long as this pressure is maintained, the " triple valve " remains set so that the reservoir *R* under the car communicates directly with the train pipe, and the brake cylinder *C* is shut off from any pressure. If the pressure in the train pipe is diminished either purposely by the engineer or thru an accident to the pipe, the triple valve sets itself so that air from the reservoir enters the brake cylinder and, forcing the piston over, sets the brakes. When a suitable pressure is re-established in the train pipe, the triple valve resets itself in such a way that the air is released from the brake cylinder and the spring then pushes the brakes back away from the wheels. The construction of the triple valve itself is complicated and need not be discussed here.

REVIEW.

1. Describe the mercury barometer.
2. State Boyle's Law.
3. Under what conditions would wood float in air?
4. Describe and explain the operation of the siphon.
5. How do air brakes work?

CHAPTER III

THE MOLECULAR THEORY OF MATTER

1. Molecules.—In this chapter we shall discuss a number of phenomena connected with gases and liquids, yet of an entirely different type from those we have been considering so far. These phenomena include among others those of diffusion, evaporation, dew and cloud formations, soap bubbles, oil films and the drying actions of blotting and towels. This list seems to cover a number of very dissimilar actions, yet the explanation of all is based on one set of facts, namely those of the *Molecular Theory of Matter.*

This theory has already been referred to in the introduction where we said that all forms of matter, that is, all objects and materials, have a very fine granular constitution, the granules being called molecules. These molecules, which are supposed to be constantly moving about in a perfectly irregular way, are entirely too small to be perceived by even the most powerful microscope. In fact, they are so small that it would be necessary to put more than 1,000 of them side by side before a speck would be formed sufficiently large to be seen with a microscope. The number of them contained in a cubic centimeter of air, at ordinary air pressure, is about 27 billion billion (27×10^{18}).

These molecules are not closely packed together. Even in dense solids there are spaces in which the molecules can move about, while in gases only a very small fraction of the total volume is made up of the molecules

—so that the spaces between are, compared with the size of the molecules, very large. It is because of this that gases have such small densities. The water that we get from the condensation of a mass of steam occupies only $\frac{1}{1600}$ of the volume of the steam, and when air is reduced to a liquid, the liquid has only $\frac{1}{800}$ of the volume of the air. These figures serve to show how small a part of the total volume of a gas is taken up by the material particles which compose it.

It must not be thought however that the space thru which the molecules of a gas at ordinary pressures move before striking other molecules is large as judged by our ordinary ideas of largeness. In air at 76 cm. pressure the average distance moved by a molecule between collisions, that is its "free path," is only about one ten-thousandth of a millimeter (.0001 mm.). But even this distance is more than fifty times the diameter of one of the molecules.

2. Molecular Motion.—Since molecules are so small that they cannot be observed directly, it is of course clear that our knowledge of their existence depends entirely on indirect evidence. There are many facts of ordinary observation which indicate that matter must be made up of fine moving particles with spaces between. Consider for example sugar dissolving in water or the odor of flowers diffusing itself thru a room. Or let us take two bottles, one filled with gaseous carbon dioxide and the other with hydrogen. If we invert the bottle containing hydrogen and place it mouth to mouth on top of the dioxide bottle and leave the bottles in that position for a few hours, we shall find at the end of that time that both bottles are filled with a uniform mixture of the two gases. As the carbon dioxide in the lower bottle is

twenty-two times as heavy as the hydrogen in the upper bottle, it is very difficult to explain this diffusion except as the result of motions of very fine particles of the two gases—in other words as the result of molecular motion.

Another indication that gas molecules are in rapid motion is found in the fact that gases are capable of indefinite expansion; that is to say, no matter how much we increase the space in which a body of gas is contained, the gas instantly expands and fills the whole space. This is what happens when the piston is raised in the cylinder of the air pump described in the last chapter.

The velocities with which the molecules of various gases move have been measured and have been found to be very high. Air molecules under ordinary conditions travel 445 meters in a second and hydrogen molecules travel 1700 meters (just about one mile) in a second. The velocity of a rifle bullet is somewhat less than half that of a hydrogen molecule. It is evident that when a gas is contained in a closed vessel the molecules striking with these enormous speeds against the walls will exert a considerable force on the walls. The total amount of this force will be proportional to the number of molecules striking—doubling the number of molecules will double the force and so on. From this consideration we see at once that if we put into a closed vessel containing a gas a sufficient number of extra molecules to double the number of molecules in it, thus doubling the density, the pressures acting on the walls of the vessel will be doubled. It is evident that in all cases the pressure exerted by the inclosed gas will be directly proportional to its density, which is exactly the statement we made for Boyle's Law in the preceding chapter.

We shall now consider a very striking illustration of these principles. A cylindrical cup of unglazed porous

earthenware is plugged at the open end with a cork thru which a glass tube passes. The end of the glass tube dips below the surface of some colored water in a dish and a large glass bell jar, left open at the bottom, is inverted over the porous cylinder. Illuminating gas, which always contains a large amount of hydrogen, is allowed to flow up from a tube into the bell jar. Bubbles immediately begin to issue from the end of the glass tube showing an increase of pressure inside the porous cylinder. If now the bell jar is removed, the colored water will rise inside the glass tube showing that the pressure inside the cylinder has become less than it was before.

The reasons for these pressure changes are these: In the first stage of the experiment, there is the same number of molecules per cubic centimeter of hydrogen outside the porous cup as there are molecules per cubic centimeter of air inside the cylinder. But the hydrogen molecules move four times as fast as the air molecules; there will consequently be four times as many impacts of hydrogen molecules on the outside of the porous cup in one second as there are impacts of air molecules on the inside; and since a certain proportion of these molecules pass thru the pores of the cup, there will be roughly four times as many molecules entering the cup as leaving it in the same time. The pressure will therefore rise and this rise is evidenced by the escape of bubbles from the end of the tube.

In the second stage of the experiment, after the bell jar is removed, there will be hydrogen and air inside the cylinder and air outside. Under these reversed conditions there will be more molecules escaping in a second than enter in the same time and the pressure inside will

therefore fall so that the colored liquid will rise in the tube.

3. Effect of Heat on Rate of Motion.—Let us now consider briefly the effect of heat on the motion of the molecules. If we take a glass bulb containing air with a long U-shaped tube containing mercury, attached to it and heat the air in the bulb, we shall find that the mercury is at once depressed in the leg of the tube next to the bulb, showing an increase in pressure due to the heating. This increase is very easily explained by the molecular theory. According to a clause of that theory not previously stated, the effect of applying heat to any body is to increase the rate of motion of the molecules—in the case of solids this causes the solid to expand—in the case of liquids, it causes some of the molecules to fly out of the liquid, thus producing evaporation, and in the case of gases, it increases the volume or—if there is resistance to expansion—the pressure of the gas. We must keep this important development of the molecular theory clearly before us, namely, that heat applied to matter in any of its forms makes the molecules move faster. The simple experiment with the glass bulb described in the foregoing illustrates this effect in a gas. For our present purposes however we are more interested in this same effect in liquids.

4. Evaporation and Condensation.— If water is left in an open dish, we know that the dish will in time become quite dry. This results from what we call "evaporation." The process is easily explained by supposing that the molecules of the liquid in their natural motions, in which some move much faster than others, shoot off one by one into the air until all are gone. It is clear that applying heat to the liquid ought to increase

the rate of evaporation since the heat will increase the speed of movement of the molecules, so that more of them will fly out in a given time. And it is common experience that this increase in rate of evaporation with rise in temperature does take place. Further, the dish becomes dry only if it is left in the open where currents of air move over it. For in that case the water molecules are carried away as they emerge from the water surface and before they can by any chance return to that surface. If now we cover the dish with a bell jar, the conditions will be very different. The molecules which fly out will then move about in the closed space inside the bell jar and many of them will strike into the liquid surface again and stay there. The molecules moving about above the surface of the liquid constitute the " vapor " of the liquid. A vapor is different from a gas only because it can easily be reduced again to the liquid form. In fact a vapor is a gas near its point of liquefaction. As the number of molecules flying about in the vapor increase, the number returning to the liquid will also increase until finally as many will return in a second as emerge in the same time. When this stage is reached, no further net loss will take place from the liquid. The vapor over the liquid is then said to be " saturated," its density, which depends upon the number of molecules of the liquid in it in each unit volume, does not change with time if the temperature is steady. If however the temperature of the liquid be raised, the molecules in the liquid move faster, more come out in a unit time, more will be moving about in the space above and finally more will return in a unit time. In the new state of balance between the numbers coming off and the numbers returning there will be more molecules in a unit volume of the vapor than there were before—in other words the

density of the saturated vapor will be increased as a result of the increased temperature of the liquid.

We have then, these facts: When a liquid evaporates into a closed space, the evaporation goes on until a saturated vapor is formed in which the number of molecules returning to the liquid per second equals the number coming off from the liquid in the same time. The density of this saturated vapor depends on the temperature, being higher for high temperatures than for low. What will happen when a dish of water is placed in a closed space containing saturated water vapor depends on the relative temperatures of the water and the vapor. If the water is warmer than the vapor, some of the water will evaporate into the space and the density of the vapor will increase until it is saturated at the higher temperature, that is until the number of molecules coming out of the liquid in a unit of time equals the number entering it. When this point is reached, the temperature of both liquid and vapor will be the same. If, on the other hand, the vapor is warmer than the liquid, some water will condense out of the vapor and join the water surface; and when water and vapor have come to the same temperature, the vapor will be saturated at that temperature, that is, as we have seen, the number of molecules going into a unit area of the water surface will be equal to the number coming out in the same time. Finally, if we take a volume of water vapor which is not saturated at the temperature of its surroundings and steadily lower its temperature, we will soon bring it to a saturated state. If we lower the temperature still further, water will begin to condense out of the vapor on to the walls of the container at such a rate that for any given temperature the vapor will always be saturated.

5. Clouds, Rain, Dew, etc.— The facts here brought out enable us to explain a great many ordinary happenings in nature. On account of the wide distribution of water the atmosphere always contains a large amount of water vapor which in the immediate neighborhood of large bodies of water is likely to be saturated or nearly so. Water vapor saturated at 32° F.—the melting point of ice—exerts a pressure equal to the weight of a column of mercury of unit cross section 4.6 millimeters high. At 70° F. the pressure of the water vapor is equal to that exerted by a mercury column 18.5 millimeters high, while at 104° F., its pressure equals that of about 55. millimeters of mercury. A cubic meter of saturated water vapor contains 4.8 grams of water at 32° and about 50.6 grams at 104°. These figures show that if a cubic meter (1,000,000 cubic centimeters) of air saturated at 104° F.—a temperature which might be attained on the surface of a quiet pool in hot summer sunlight—be lowered to 32° about 45 grams of water will be precipitated from it. This change in temperature might easily be brought about by a vertical rise of a few thousand feet. This rise would naturally result in accordance with Archimedes' principle from the lowered density of the superheated air over the pool. The condensed water would form a multitude of small globules which would produce a *cloud.* If these drops grew by further condensation to a sufficient weight they would fall as *rain.* The vapor remaining over the surface of the pool when the sun went down would cool and precipitate its water content to form a low-lying *mist* or *fog.* If the vapor spread over the country-side and came in contact with the cool grass after sundown, the water would precipitate as *dew.* If the upper atmosphere was sufficiently cold, the moisture deposited in the cloud would form in

the process of condensation into very beautiful minute crystals and would come down as *snow* rather than as rain. Under certain special conditions in the atmosphere in which layers of rain and snow clouds alternated, *hail stones* would fall consisting of concentric layers of snow and ice.

6. Evaporation and Temperature.—If we put one thermometer into some ether in a deep narrow tube and another into ether in a wide shallow dish, we find the temperature in the shallow dish to be somewhat lower than in the tube. This is what we might expect since the evaporation from the broad exposed surface is necessarily more lively than from the surface in the narrow tube. Evaporation from a liquid surface always lowers the temperature of the liquid. The molecular theory indicates that this effect should be expected, since in evaporation the molecules which fly out are always those with velocities somewhat above the average, so that the average velocity of those left behind is less than it was before, which means that the temperature is lower. If the rate of evaporation from the surface is high, the cooling effect may be very marked. Drinking water is kept cool in hot countries by putting it in goat-skins which are somewhat porous and hanging the skins up in a current of air where evaporation can take place freely from the moist surface. The surface of the human body moistened with perspiration is cooled in a similar way by a current of dry air. It will be seen that this cooling effect cannot take place if the current of air contains saturated water vapor, for under these conditions no evaporation can take place. The effect of high temperatures on the human system therefore depends to a great extent on the amount of moisture in the air, or as we say,

on the *humidity* of the air. In the summer time we can bear high temperatures much more comfortably when the air is relatively dry than when it is moist. From this point of view, it is important to know the relation between the amount of moisture actually in the air and the total amount which it can take up at the particular temperature, because on this relation will depend the rate of evaporation from water in contact with the air. This ratio is called the *Relative Humidity*. It is usually stated in weather reports during hot weather and is quite as important as the temperature in determining the effect on us of the hot weather.

7. Determining the Relative Humidity.—The relative humidity can be determined in this way. Ice is added to water contained in a brightly polished nickle-plated can and stirred round with a thermometer. When the temperature of the water vapor around the can is reduced to its saturation temperature, moisture will begin to form on the polished sides of the metal vessel where it can be detected as a thin film. At this instant the thermometer is read and the temperature at which the water vapor in the air is saturated thus determined. This is known as the *dewpoint*. Suppose this temperature is found to be 15° Centigrade. We now look up the pressure of saturated water vapor at 15° in a table such as can be found in any scientific reference book. We find this pressure to be equal to that of 12.7 millimeters of mercury. This represents the pressure of the water vapor actually in the air at the time the measurement is made. On a second thermometer we now read the temperature of the air. Suppose this temperature to be 25°. In the same table we find that *saturated* water vapor at 25° will exert a pressure of 23.5 millimeters of

mercury. It is therefore plain that the air contains at this time $\frac{12.7}{23.5}$ or a little more than one-half as much vapor as it can hold. The relative humidity in this case is .54 or 54%. On very uncomfortable and exhausting days in summer the relative humidity lies between 80% and 90%. For the best and most healthful conditions it should lie between 50% and 60%.

8. Evaporation in a Vacuum.—Before leaving this section of our subject one other important and rather peculiar point must be referred to. Experiments show that the amount of a liquid which will evaporate into a closed space of a given volume is the same when that space is a vacuum as when it is filled with air. The evaporation takes place more slowly in the second case on account of the great number of collisions between the water molecules and the air molecules but the total amount of liquid evaporated is the same in both cases. At first sight this seems rather peculiar. The reason for it lies in the fact already referred to that the spaces between the molecules of gases constitute by far the greater part of the volume of a gas. The experimental facts above stated show further that if a space contains a number of different vapors each of these vapors will exert the same pressure as if it were alone in the space and the total pressure will equal the sum of the pressures of the different vapors. When the pressure of the atmosphere is measured, the value includes the pressure of the water vapor in the air as well as the pressure of the air itself. Moreover, the phenomena described above are not dependent on any absorptive power possessed by air similar to that of a sponge for water. They take place in the absence of air more rapidly and completely than in its presence.

9. Molecular Forces in Liquids.— So far in this chapter we have been studying mainly molecular motions in matter in the gaseous state. These motions, for the most part, show a tendency of the molecules to fly away from one another. But there are also certain forces acting between molecules of certain kinds tending to hold them together. The single fact that gases are indefinitely expansible indicates that the attractive forces between the molecules of gases are very small if there are any at all. The molecules of solids on the other hand hang together very tenaciously. Thus it requires a weight of about eight tons to pull in two a steel rod 1 centimeter in diameter.

In liquids we have an intermediate condition. Altho liquids hang together so that they can be poured from one vessel to another in a continuous stream, yet they apparently possess very little longitudinal strength. We are for this reason likely to underestimate the strength of the molecular forces in liquids. However if we take a sheet of glass with an area of one square foot and after providing it with a handle by waxing a block of wood to its center let it down horizontally on to a surface of water so that the water wets it, we shall be surprised to find it necessary to exert a very considerable force to pull the plate free from the water surface again. Since, after removal, the plate will have a film of water adhering to it—that is, will be wet,—it is clear that the force used was expended actually in tearing the liquid apart—that is in overcoming the molecular attractive forces in the liquid.

Let us consider the way in which these attractive forces act at different places in a body of liquid as at *P* and *Q* Fig. 20. At *P* the molecules will be pulled

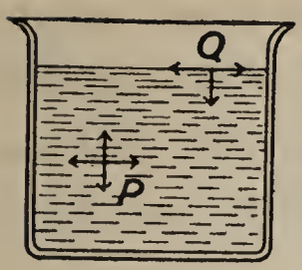

Fig. 20.

equally in all directions—there will consequently be no resultant force acting to affect the motion of the molecules in any way. At Q however or at any other point in the liquid surface, the forces will be unbalanced with a resultant force directed inward. These unbalanced forces produce an effect in the surface film exactly like that which would be produced if the surface were elastic and contractile. The mass of liquid has a tendency to draw itself together into the most compact possible shape, that is into the shape which has the least surface for the greatest volume. This shape is that of a sphere. The liquid in the vessel does not take the shape of a sphere because the forces due to its weight are much greater than the forces exerted by its surface film. If we could in any way nullify the force of gravity we ought to expect the liquid to take a spherical form.

We can produce this effect in the case of lubricating oil by making up a solution of alcohol and water of the same density as the oil and then introducing a large drop of the oil below the surface of the alcohol-water mixture thru a glass tube. The oil will immediately take up a perfectly spherical form. Since in these conditions the buoyant effect on the oil drop is exactly equal to its weight, the influence of gravity is nullified and the observed form of the globule is due entirely to the action of the tension in the liquid surface. Whenever the force due to gravity acting on the mass of liquid is small as is the case with very small drops, the shape taken up will be more or less spherical—for example small droplets of mercury scattered on a table are almost exactly spherical as are also very fine droplets of water. This contractile

tendency of liquid surfaces can be measured and when so measured is called the " surface tension " of the liquid. It is different for different liquids. Thus, the surface tensions of alcohol and of kerosene are about one-third the surface tension of water.

The existence of this contractile tendency in liquid films is illustrated in blowing soap bubbles from a pipe. If the inside air pressure is released while the bubble is still attached to the pipe, the bubble will immediately contract into a flat sheet over the end of the pipe.

10. Cohesion and Adhesion.— When the finger is dipped into water and then withdrawn, it is found to be wet, but when withdrawn from mercury, it remains perfectly dry. Whether or not a liquid wets the finger depends therefore on whether the attractive force of the material for itself—the so-called *cohesive* force of the liquid—is greater or less than the attractive force of the liquid for the skin of the finger—the so-called *adhesive* force. These terms, cohesive and adhesive, are of general application—the term cohesive being applied to attractive forces between molecules of the same kind and the term adhesive to forces between molecules of different kinds. Whenever a solid is dipped into a liquid and comes out wet, it is evident that the adhesive forces are greater than the cohesive, while if the solid comes out dry, the cohesive forces must be superior to the adhesive ones.

Let us consider somewhat further the force actions in the surface of a body of water where it meets the wall of a glass containing vessel. Since the effects due to gravity are very small compared with the effects due to cohesion and adhesion in all cases where the mass of liquid considered is small, we shall in this discussion

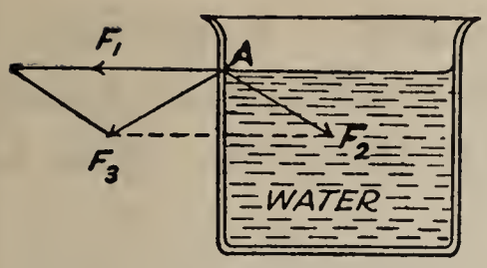

Fig. 21.

leave out of consideration the weight of the water, being concerned only with a very small quantity of the liquid where its surface touches the glass.

The conditions are represented in Fig. 21. F_1 shows the direction of the force on the water molecules due to the attraction of the glass. F_2 shows the force acting on the same molecules due to the bulk of the water. Since in water the cohesive force is less than the adhesive force for glass, F_2 is represented by a line shorter than F_1. The resultant force acting can be represented by F_3 in magnitude and direction. That this is true will be definitely proven in the next chapter—just now we will take the statement without proof.

We must now recall a proof given in the chapter on hydrostatics that the free surface of a liquid is always horizontal. This proof can be extended and stated in this form: that the surface of a liquid is always perpendicular to the line of the resultant force acting on it; if the force of gravity is the only force acting, the surface will be horizontal since a horizontal surface is defined as one perpendicular to a plumb line. Applying this principle to the case under consideration it is plain that the water surface should be curved up at its line of contact with the glass walls in order to be perpendicular to F_3 as shown at A. It is in actual fact, so curved.

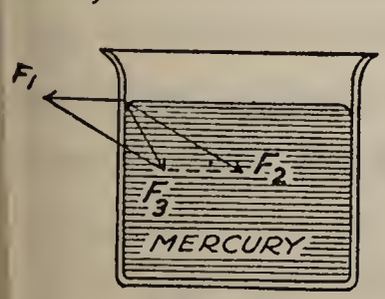

Fig. 22.

Consider now Fig. 22, in which are shown the conditions existing when mercury is contained in glass. Here F_1 the adhesive force is small, F_2 the cohesive force is large and F_3 the re-

sultant is consequently directed inwards. According to the rule just given the liquid surface in this case should be curved downward where it meets the glass. This is in fact found to be true.

11. Capillary Elevation and Depression.—Since the liquid surfaces here considered have a tendency to contract as if they were elastic, it is clear that a small lifting force ought to be exerted on the water in Fig. 21 and a small depressing force ought to act on the mercury in Fig. 22. The existence of these forces can be rendered evident by reducing the weight of liquid acted on by cutting down the diameters of the vessels. It is necessary to make the diameters very small before the effect becomes marked—the vessel must, in fact, be reduced to a fine tube. Such tubes are called capillary or hair-like tubes. When one of these tubes is dipped into water, the water at once rises to a greater or less height depending on the internal diameter of the tube. This is known as "capillary elevation." On the other hand, when the tube is dipped into mercury the mercury surface inside the tube will stand lower than the surface outside by an amount depending on the diameter of the tube. This is known as "capillary depression." These effects, which are of considerable practical importance, are easily explained by the principles which we have just now been discussing.

When the tube is dipped into a liquid which wets it, the liquid surface inside the tube will be *concave* upwards as shown in *A*, Fig. 23. The surface will immediately contract in order to flatten itself and as soon as it is flat it will at once rise again around the edges and so on, thus rising in the tube and drawing after it a column of the liquid until the weight of that column

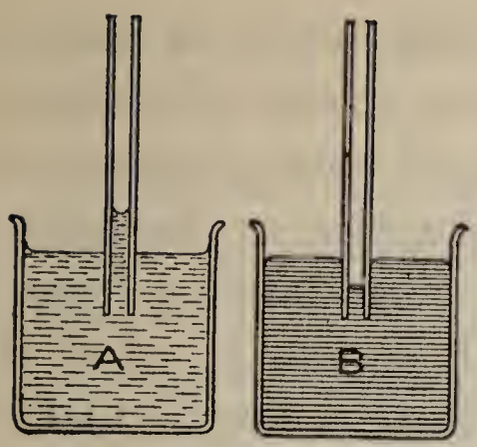

Fig. 23.

exactly balances the contractile tendency of the surface. When the tube is dipped into a liquid which does not wet it, the liquid surface inside will be *convex* upwards and in contracting will push the liquid down until the upward pressure due to the difference of level inside and out equals the contractile force of the surface (*B*, Fig. 23).

There are many familiar illustrations of capillary effects. The flow of oil upward thru a wick is due to this effect as is also the complete wetting of a towel in case one corner is left in a dish of water.

These same principles enable us to explain how certain kinds of insects walk or skate over the surface of water. The water does not wet the feet of the insect which therefore make small depressions in the surface in which the surface film is concave upward. The contractile forces in these depressions suffice to hold the insect up. A striking experiment showing the same effect can be made with a sewing needle. If a needle is laid carefully on a water surface it will remain floating on the surface in spite of the fact that it is eight times as dense as the water. The water does not wet the needle and the water surface around the needle is therefore concave upward as in Fig. 24. Since the contractile tendency of the film in the depression is greater than the weight of the needle, the needle cannot sink. If the film is ruptured in any way, the needle will of course go thru and sink directly to the bottom.

Fig. 24.

12. Molecular Forces in Solids.— We close this chapter with a

brief study of the action of molecular forces in solids. In solids the cohesive forces are very great. We have already noted that it takes a force of about eight tons to pull in two a steel rod one centimeter in diameter. The *tenacity* or *tensile strength* of a substance is measured by the force necessary to pull in two a rod of it of a certain cross area. Tensile strength is usually measured in kilograms per square millimeter. In these units the tensile strength of lead is about 2; of steel piano wire about 200; of brass wire about 38; of hard drawn copper about 45; of oak about 8; of catgut about 42; and of quartz fibers about 100.

If we take a wire of any material and apply loads to it, considerably less than the load necessary to break it, we find that stretches or elongations are produced by the different loads which are proportional to the loads —that is doubling the load doubles the stretch and so on. We also find that when the load is removed, the wire will contract to the length which it had before stretching. This tendency of the body to return to its original volume and shape after being distorted is known as *elasticity.* If, however, the applied load exceeds a certain definite amount, the wire when released will remain permanently stretched. This definite limiting tension is known as the *limit of perfect elasticity* for the substance. It is very different in different materials. A highly elastic substance is one which returns rapidly and accurately to its former shape and size after being stretched. From this definition it is plain that substances like steel, glass and ivory have much higher elasticities, than elastic rubber bands and such things which, altho they can be stretched to a great length, return to their original states only very slowly and imperfectly. Billiard balls are consequently made of ivory

rather than of rubber and vehicle springs are ordinarily made of steel.

13. Hooke's Law of Elasticity.— Robert Hooke (1635-1703) first showed by accurate experiments that if the limits of perfect elasticity are not exceeded, the deformations produced in elastic bodies are proportional to the loads applied. The full statement of Hooke's Law, as it is called, is as follows: "Within the limits of perfect elasticity, elastic deformations of any kind, twists, bends or stretches, are directly proportional to the forces producing them." This law has many important applications in physics which will be referred to as they arise in the course of our studies. Hooke first published this law as an anagram at the end of a book which he issued in 1676. The anagram read *veiiosstttnuu*. He gave the translation, "Ut tensio ut vis," some two years later. This Latin phrase means "The extension is proportional to the force acting." Hooke's exact expression of his law in English is "The power of any spring is in the same proportion with the tension thereof—that is, if one power stretch or bend it one space, two will bend it two and three will bend it three, and so forward."

We have now completed our discussion of the forces acting between the molecules of solids, liquids and gases. We must next turn our attention to the study of forces acting between large bodies—that is to say between bodies large in comparison to molecules.

REVIEW.

1. State the molecular theory of matter.
2. Why does evaporation from a liquid surface always lower the temperature of the liquid?
3. Define (a) cohesion and (b) adhesion.
4. State Hooke's Law.

CHAPTER IV

COMPOSITION AND RESOLUTION OF FORCES

1. Force Units.—It will be remembered that in the Introduction, we defined a force as an action in the nature of a push or a pull between two bodies which tended to change the motions of the bodies. The unit of force that we shall make use of in this chapter has the same name as the unit of mass. It is necessary therefore to take particular care to avoid confusion of the two units. The unit of mass in the metric system is the *gram,* that being the mass of a cubic centimeter of pure water at 4° Centigrade—the unit of force is the earth's pull on a gram of mass. It is called the *gram of force.* While the mass of a gram is, of course, definitely the same at all times and places, experiment shows that the earth's pull on a gram of mass varies as we move from the equator toward the poles and also as we ascend a high mountain. In other words, the gram of force is not constant—it varies. At the equator it is about 5 parts in 1000 *less* than it is at the north pole. This is due partly to the rotation of the earth and partly to the flattening of the earth at the poles, both factors affecting the earth's pull on a body in ways which will be explained later.

In the English system, the unit of mass is the *pound* and the unit of force is the earth's pull on a pound of mass. This pull, called the *pound force,* varies of course from place to place, the same as does the gram force. There are other fundamental force units besides the gram force and the pound force, but for the present we

shall make use of only these two. In order to avoid confusion with the units of mass we shall make a practice of using the expressions "gram force" and "pound force" in speaking of force units.

2. Measurement and Comparison of Forces.—A usual way of comparing the relative values or magnitudes of forces is to allow them to act successively on a coiled spiral spring as in a spring balance. According to Hooke's Law a double force will produce a double extension and so proportionately. A spring can easily be calibrated, that is graduated, in successive units, by hanging known weights to it and marking off the extensions on a fixed scale. When so graduated it can be used directly for the measurement and comparison of forces.

A force, being in the nature of a push or pull, has always a definite direction and sense—a definite magnitude in grams force or pounds force—and, when acting on a body, a definite point of application. We can always represent a force by a straight line drawn in a definite direction (with an arrow head on it to show its sense) and of a definite length proportional to the magnitude of the force in grams force or pounds force. The beginning of this line shows the point of application of the force. The word "sense" is somewhat unusual in this connection and ought therefore to be explained. A railroad track laid north and south has a definite direction, north-south, but no sense. The motion of a train running on the track has a direction north-south no matter which way the train is moving. If the train is going north, the motion is in a northerly *sense,* while if it is going south, the motion is in a southerly *sense.* It is clear that in representing the motion of the train by a line, the direction of the line represents the direction

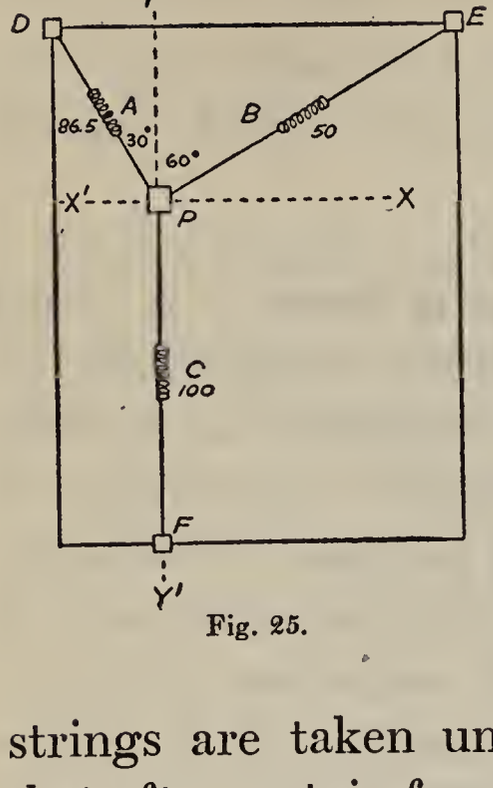

Fig. 25.

of the motion but an arrow head is needed on the line to indicate its sense.

3. Resultants and Equilibrants.—Let us now consider the experimental arrangement shown in Fig. 25. Three spiral springs, *A*, *B* and *C*, graduated in grams force are attached to a small plate *P* by three strings. From the other ends of the springs, strings are taken under the clamps *D*, *E* and *F*, so that after certain forces have been applied to the springs, the strings can be clamped and things kept stationary. Let the angle of *PD* with *Y* be 30° and of *PE* with *Y* be 60°. Let *PF* lie along the line YY^1. Then when *C* reads 100 grams force, *B* must read 50 grams force and *A* 86.5 grams force in order for *P* to remain stationary. It is evident that the effect produced by forces *A* and *B* acting together could be produced by a single force alone, say *G*, of 100 units acting along *PY* directly opposite to force *C* for in that case *P* would still remain stationary, *G* and *C* being equal and directly opposed, the resultant force acting on *P* clearly being zero. Force *G* can be called the *resultant* of forces *A* and *B*—a *resultant of two forces* being defined as the single force which will produce the same effect on a body as is produced by the action of the two forces together. Again, if *A* and *B* were acting on *P*, and *C* did not act, *P* would move. The force *C* therefore neutralizes the combined effect of *A* and *B* and prevents the motion of *P*. *C* is consequently called the *equilibrant* of *A* and *B*—the *equilibrant of a force* or

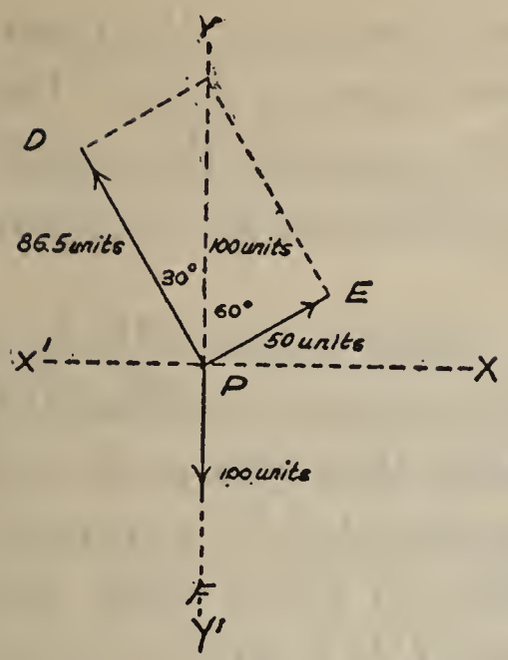

Fig. 26.

forces being the single force which will just prevent the motion which the given forces tend to produce. The equilibrant of a system of forces is always equal and opposite to the resultant of the same system.

We can represent the forces in this experiment by lines and thus make a "graphic construction." Fig. 26 represents such a construction. To make this drawing, put in the reference lines XX^1 and YY^1 then lay down PE 50 units long (say 50 millimeters, representing a gram force by a millimeter of line) at an angle of 60° to the right of YY^1. This angle can be measured with a protractor. Then draw PD 86.5 units long at an angle of 30° on the other side of YY^1. Finally put PF 100 units long on the line YY^1 and put in arrow heads as shown. If now a parallelogram is constructed on PD and PE as sides, it will be found that PY is the diagonal of that parallelogram and furthermore that PY measures exactly 100 units in length. The diagonal of the parallelogram constructed on the lines representing forces A and B as sides therefore represents the *resultant* of A and B. We see further that the line PY^1, equal in length and opposite in sense to this diagonal, represents the *equilibrant* of A and B.

A general proposition, true for any two forces lying at any angle whatever, can be stated as follows: *To find the single force which will produce the same effect as two given forces, that is to find the resultant of two forces,* lay the forces out graphically representing them by lines

of the proper lengths in the proper directions; then construct a parallelogram on these two lines as sides. The diagonal of this parallelogram will represent the resultant of the two forces both in magnitude and in direction.

4. Components.—Let us now suppose that *PF* is a girder in a bridge truss and that it sustains a pull of 100 tons force. Let us suppose further that this pull is to be neutralized or in other words supported by two girders along *PD* and *PE*. In order to know what size girders will be required on *PD* and *PE* it will be necessary to calculate the magnitude of the pulls along these girders due to the given pull of 100 tons force on *PF*. It is in the first place evident that a force equal to that in *PF* and directly opposite to it in direction would serve the necessary purpose of support. This would be the force *G* referred to in the last paragraph. The problem therefore reduces itself to this: To find the two forces acting along *PD* and *PE* which will produce the same effect as force *G* along PY^1. This is known as determining the *components* of the single force *G* along the two given lines *PD* and *PE*. How to make the solution graphically is clear from what has already been said about the parallelogram of forces. On the line representing the single force *G* as a diagonal construct a parallelogram with two sides in the two given directions *PD* and *PE*. The lengths of these sides after the parallelogram is finished give the magnitudes of the required components. The application of this simple rule to the problem in hand shows that the component along *PD* must be 86.5 tons—while that along *PE* must be 50 tons. This problem is of course the inverse of that treated in the preceding paragraph.

We have now developed two graphical methods, the

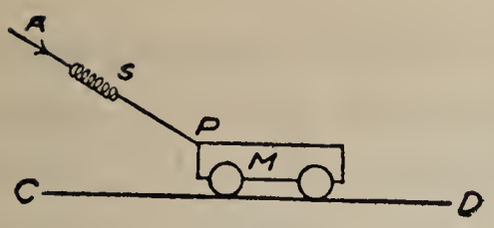

Fig. 27.

first for finding the resultant of any two given forces acting at one and the same point in given directions, the second for finding the components of any given force along two given lines. These methods can be applied to the solution of many different problems. Let us consider two or three specific cases.

5. Force Actions: Horizontal Plane.—First take the case of a car M (Fig. 27) being shoved along a level track by a man pushing on the handle AP. Suppose that a compressible graduated spring S in the handle shows the applied force to be 100 pounds force. It is evident that only a part of this force is effective in moving the car along the track, since a certain proportion of it merely pushes the car down on the track without tending to move it. It is of course of practical importance to know what is the value of the force effective in moving the car: that is to know the *component* of the force AP along the direction of the track CD. We can calculate this component and at the same time find the component pushing the car down on to the track by applying our parallelogram rule. This is done in Fig. 28. Draw A^1P^1 100 units long at the proper angle, that is parallel to AP in Fig. 27. Draw C^1D^1 parallel to CD and put in P^1E^1 perpendicular to C^1D^1. These two lines C^1D^1 and P^1E^1 are in the direction of the two components that we wish to determine—one along the track, the other perpendicular to it. Now, according to our rule, complete the parallelogram on the diagonal A^1P^1. The sides of the parallelo-

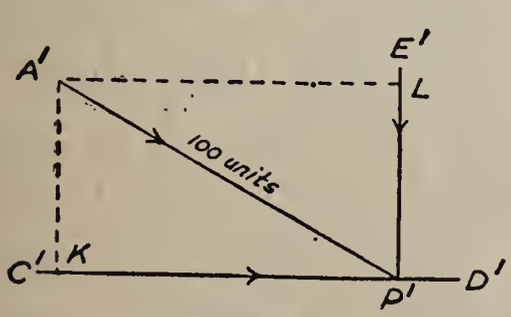

Fig. 28.

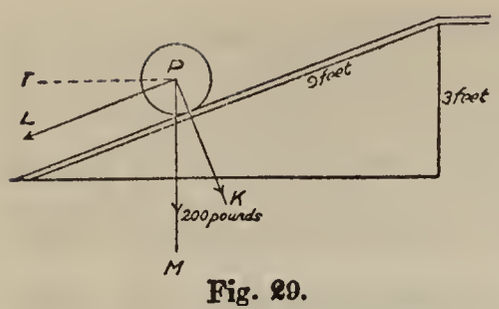

Fig. 29.

gram KP^1 and LP^1 when measured in the same scale of units used in laying out $A^1P^1 = 100$ units give the effective forces acting in the two directions—that is to say the components of the single force A^1P^1 in the two directions. With the conditions shown in Fig. 27 a force of 92.5 pounds is available out of the man's push of 100 pounds force to move the car along the track.

6. Force Actions: Inclined Plane.— Next consider the case of a barrel P weighing 200 pounds being rolled up a plank 9 feet long into a wagon the bed of which is 3 feet above the ground, Fig. 29. The question is to find the force which must be applied parallel to the plank in order to keep the barrel from rolling down. The active force is the weight of the barrel 200 pounds force acting vertically down toward the center of the earth along the line PM. This force can easily be resolved into two components one PL, giving the force along the plank tending to make the barrel roll down, and one PK perpendicular to the plank which produces no effect excepting to press on the plank itself. In order to keep the barrel from rolling down it is merely necessary to apply a force along the plank equal and opposite to the component PL. To make the solution which is represented in Fig. 30 first draw P^1M^1 parallel to PM and 200 units long—then put in the lines P^1L^1 and P^1K^1 parallel respectively to PL and PK. Next apply the parallelogram rule as before, con-

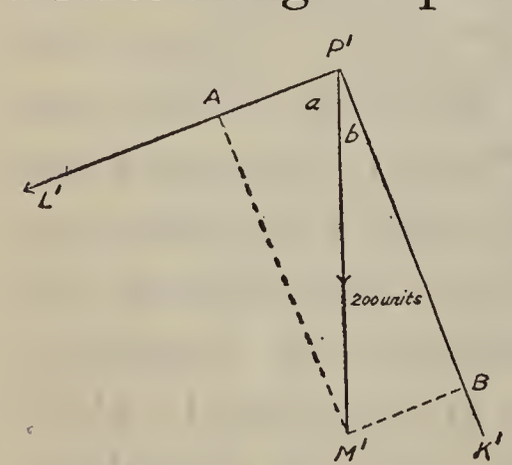

Fig. 30.

structing the parallelogram P^1AM^1B on P^1M^1 as a diagonal. The lengths of the sides P^1A and P^1B when measured on the same scale used to measure P^1M^1 will give the component forces sought. We thus find that 66.6 pounds force must be applied along the line PL in order to prevent the 200 pound barrel from rolling down. It will be noted that the length and height of the plank enable us to plot the angles a and b in Fig. 30 and thus affect the solution which would be different for different lengths or heights. It is suggested that the reader determine by the same method the force which must be applied parallel to the ground, that is along the line PT Fig. 29, to hold the barrel stationary.

7. Newton's Three Laws.— We shall next turn our attention to a series of remarkable fundamental generalizations or laws dealing with force and motion made by the famous Sir Isaac Newton (1642-1727).

8. Inertia.—The first of these laws deals with a tendency of matter with which we are all familiar. We know that if a person steps from a rapidly moving car to the ground he is likely to be thrown down. We also know that passengers in a stationary car feel a very distinct jerk if the car starts suddenly into motion. These and many similar familiar facts show that when bodies are in motion they tend to continue to move, and that when they are at rest they tend to remain at rest. Newton perceived that this tendency was a general property possessed by all matter. He called it *inertia.* Inertia is accurately defined as *that property possessed by all matter of resisting any change in its state of motion,* either attempts to stop it if moving or to start it if at rest, or attempts in any way to change either the amount or direction of its motion. It is always found necessary

to apply a force to a body in order to change its state of motion. Newton's first law covers this fact. Stated in Newton's own words the law reads:

Every body perseveres in its state of rest, or of uniform motion in a right (straight) line unless it is compelled to change that state by forces impressed thereon.

The statement that all matter possesses inertia is of course equivalent to Newton's first law provided inertia is adequately defined. It will be seen that in accordance with this law a body once started into motion will continue to move forever in a straight line unless some force acts to stop it. These conditions cannot be reproduced on the earth because it is impossible to get rid entirely of frictional forces in any machinery. There is always therefore some force acting to stop any motion which we may start. The planets, however, moving thru empty space apparently meet with no resisting forces, and thruout the centuries during which observations have been made, no lessening of the speeds of the planets has ever been detected.

9. Momentum.—Newton's second law treats of the "amount or quantity of motion" possessed by bodies. It is necessary to get a clear idea of what this term "quantity of motion" means as Newton used it. It is clear that the effect produced on a body by the application of a force depends both on the magnitude of the force and on the mass of the body. A vigorous kick applied to a football full of air will give it a motion with considerable speed, but a kick with the same force applied to a football full of sand would scarcely move it—the speed imparted would be very slight. The "quantity of motion" possessed by the football after the kick may be measured in both cases by taking the product of

the mass of the ball into its velocity or " rate of motion." If for " quantity of motion " we use, as is customary, the single word " momentum " we can say that the momentum of any body at any given instant may be measured by the product of the numbers representing the mass of the body and its velocity at that instant. A velocity is a speed or rate of motion of a body in a certain direction. The idea of velocity is familiar to all in connection with moving trains and other objects. We measure velocity by the number of space units passed over in unit time, as, for instance, in miles per hour or feet per second or centimeters per second. We define quantity of motion or momentum, then, as follows:

$$\text{Momentum} = \text{Mass} \times \text{Velocity}.$$

If the mass is in grams and the velocity in centimeters per second, the momentum will be in metric units of momentum. There is no generally accepted name for this unit.

Before saying anything further about Newton's second law, we can by making use of the idea of momentum give a new form to Newton's first law. A body not acted upon by any external force must according to that law move in a straight line with unchanging speed—that is with " constant velocity." If the velocity is constant, the momentum must be constant, whence we can say for the first law " A moving body not acted upon by external forces has a constant momentum."

If now a moving body is acted upon by an external force, its momentum will be changed. The " time rate of change " of the momentum, that is the change in momentum in unit time, is found by dividing the total change in momentum produced by the force by the number of seconds during which the force acted. If the

original momentum of the body was mv_1 and the momentum after t seconds was mv_2 then the time rate of change of the momentum was $\frac{mv_2-mv_1}{t}$. This prepares us for the statement of Newton's second law of motion which is:

The time rate of change of momentum is proportional to the force acting and takes place in the direction in which the force acts.

This means that if a certain force produces a certain change in the momentum of a body in one second, a double force will produce a double change in the momentum in the same time and so on.

This can be put as a formula $f \propto \frac{mv_2-mv_1}{t}$ where f represents the acting force. If mv is set down equal to the change in momentum then we can write $f \propto \frac{mv}{t}$. The brief note on proportions given in the Introduction shows us that we can write this last expression in the form $\frac{ft}{mv} = k$, or $f = k\,\frac{mv}{t}$ where k is a constant. If now we define the unit of force as that force which is applied to a unit of mass for a unit of time will produce a change in velocity of one unit, then the constant k will reduce to 1, so that as long as we use this unit of force we are justified in putting down $f = \frac{mv}{t}$. In the metric system the force which will in one second produce a change in velocity of 1 cm. per second in a mass of one gram is called the *dyne*. In the English system the force which will in one second produce a change in velocity of 1 ft. per second in a mass of one pound is called the *poundal*. So far we have been using as units of force the gram force and the pound force. Experiments show that a gram force acting for one second on a gram mass gives it a velocity of 980 cm. per second and that a

pound force acting for one second on a pound mass, gives it a velocity of 32.16 feet per second. It is evident from this that the gram force equals 980 dynes and that the pound force equals 32.16 poundals.

Keeping these facts in mind the formula $f = \frac{mv}{t}$ or the equivalent expression $f = m\ (\frac{v_2 - v_1}{t})$ may be used to solve a great variety of problems. For instance, what force is necessary to impart to a mass of 20 grams a change in velocity of 10 cm. per second in each second? Here $\frac{v_2 - v_1}{t}$, which is the change in velocity in one second, is equal to 10—m equals 20—whence $f = m\ (\frac{v_2 - v_1}{t}) = 20 \times 10 = 200$ dynes. In grams force the answer is $\frac{200}{980}$ grams force. The reader when using this formula should bear carefully in mind that when the values given for the mass in grams are directly substituted for m in the formula, the force will come out in dynes.

10. Action and Reaction.— Newton's third law of motion deals with an aspect of force action to which we have already referred. It will be remembered that we have stated that a force is always in the nature of a push or pull between two bodies, acting on both bodies. Take the case of an elastic band fastened to two blocks, pulling them together, it is plain that no force can be exerted on the one block that is not equally exerted on the other. If the band is cut loose from the one block it will instantly cease pulling on the other. So for a spiral spring pushing two blocks apart. The spring must push equally on both blocks and in opposite directions. So with a bullet fired out of a gun where the expanding gases must push both on the gun and

on the bullet. It is so with all force actions. Newton expressed this general fact in his third law by saying:

To every action there is an equal and opposite reaction.

The foregoing statement will be more useful to us if we put it in a slightly different form. Since every force action is exerted equally on two bodies, it follows that whenever a body acquires momentum, some other body must acquire an equal and opposite momentum. This principle is rigidly true, but it is not always easy to see the application of it. When an apple falls from a tree a momentum is imparted to the apple but not apparently to any other body. In this case the other body is the earth, the mass of which is so great compared with that of the apple that altho the momentum given to the earth equals that given the apple, the resulting velocity of the earth's motion is entirely imperceptible. Similarly, when a man leaps off a ferry boat, he imparts to the boat a backward momentum, m_1v_1, equal to his forward momentum, mv, but the mass of the boat is so much greater than the mass of the man that the boat apparently does not move. If, however, the same man leaps ashore out of a light canoe, the canoe will glide back with considerable speed—so much so that the man is likely to fall short in his jump. In this case the mass of the man is probably greater than that of the canoe. The mathematical expression of this law which can be made use of in solving problems is:

$m_1v_1 = m_2v_2$ where m_1 and m_2 are the masses of the two bodies between which the force acts and v_1 and v_2 are the resulting changes in velocity of the first and second bodies respectively.

11. Attraction of Bodies.— Having now completed our brief study of Newton's three laws of motion we next turn to the consideration of another law of the greatest importance put forward by this same philosopher. Newton collected a great mass of data on the motions of the planets from which he reached the conclusion that forces must act between the heavenly bodies which vary inversely with the square of the distances between their centers and directly with the product of their masses. This principle was extended to apply also to bodies on the earth giving us the final form of the law:

Any two bodies in the universe attract each other with a force which is proportional directly to the product of their masses and inversely to the square of the distance between their centers.

Using the facts about proportion brought out in the Introduction, we can write for this law $f \propto \frac{mm_1}{d^2}$ where f is the force of attraction, m and m_1 the masses of the two bodies and d the distance between their centers. When we say the force varies inversely with the square of the distance we mean this: If the distance is doubled, the attraction becomes one-fourth; if the distance is trebled, the attraction becomes one-ninth and so on. This law forms the basis of many astronomical calculations and also applies to problems dealing with the weight of bodies on or near the earth's surface. It is of course clear that the attraction referred to in the law when applied to bodies near the earth's surface is the weight of the body or the "earth's pull," previously spoken of. The law indicates that the weight of a body must decrease when it is lifted above the earth's surface. The radius of the earth is about 4000 miles. If we could lift a body to a height of 4000 miles above the earth's

surface and thus double its distance from the earth's center, its weight, that is the earth's pull on it, would drop to one-fourth of the weight it had on the surface. For all ordinary heights, however, this effect is very small. A kilogram mass on top of a mountain four miles high is pulled down by a force of 998 grams force, which is only two grams less than at sea level. In all our work, therefore, we shall regard the earth's pull on a body, that is the weight of the body, to be constant and unchangeable for any ordinary variations in height at any one place on the earth's surface.

12. Accelerated Motion: Falling Bodies.—If a body is held at a height above the earth's surface, a force due to the earth's pull will act upon it. If the body is released, it will move (in this case fall freely) under the influence of that force which will remain constant during the time of fall. The force being constant will, according to Newton's second law, produce equal changes in velocity in equal times. In other words the velocity will steadily increase. Such a motion in which the velocity increases uniformly is called *uniformly accelerated motion.* This type of motion results whenever a constant force acts on an unimpeded body. All freely falling bodies move with uniformly accelerated motion. The change in velocity in a unit time is known as the *acceleration* of the motion. In any particular case this can be calculated by dividing the total change in velocity by the time during which the change took place, that is from the expression $\frac{v_2-v_1}{t}$ which is seen to equal the acceleration a.

It is interesting to note that by making use of this new symbol we can put the expression for Newton's second law, $f = \frac{m\ (v_2-v_1)}{t}$, into the more compact form $f = ma$.

It is worth our time to consider at some length the case of freely falling bodies. In the first place we must note that according to the Law of Universal Gravitation the earth's pull on a body must vary directly with the mass of the body—if the mass is doubled the earth's pull is also doubled, and so on. It follows from this that the quotient $\frac{f}{m}$ must be a constant and therefore the acceleration a is a constant since $\frac{f}{m} = a$. This means that the acceleration of all freely falling bodies is the same, irrespective of the weight of the bodies, so that if a great variety of objects of different materials, shapes and sizes be released at the same instant from the top of a tower all will reach the ground at the same instant. In practice this will not actually be true because of the resistance of the air which would impede the fall of the bodies of large area more than it would the small compact bodies. If the experiment were carried out in a vacuum or with bodies to all of which the air offered the same resistance, the time of fall would be found to be exactly the same for all of them. This fact was first experimentally demonstrated by Galileo about 1590. Prior to his time it was universally taught that " bodies fall with velocities proportional to their weights."

Galileo made a series of experimental studies of falling bodies from which he deduced certain mathematical expressions for calculating the velocity gained and the space passed over in any given time. We can readily deduce these formulæ for ourselves from the following considerations.

First, the acceleration, a, is the gain in velocity in one second. The total gain in velocity in t seconds is clearly equal to at. We can therefore write for uniformly accelerated motion $v = at$ where v is the gain in velocity

in time t. Second, velocity is defined as the space covered in one second. It is clear that if the velocity v is uniform, the space s covered in t seconds is given by the formula $s = vt$. If, however, the velocity is not uniform, the space can still be determined if we know the *average velocity* during the time t. If we represent this average velocity by a then $s = a\ t$. In the case of uniformly accelerated motion where the body starts to fall from a position of rest, we have the velocity at the beginning of the time t equal to zero while at the end of the time it equals at. The average velocity during the time t is then $\frac{o\ plus\ at}{2}$ or $\frac{at}{2}$. From this expression for the average velocity we can write $s = a\ t = \frac{at}{2} \times t = \frac{1}{2}\ at^2$. This formula $s = \frac{1}{2}\ at^2$ enables us to calculate the space covered in any given time by a freely falling body that starts from rest. In problems it is frequently desired to determine the final velocity of the falling body from the space covered. We can get a direct formula expression between these quantities by substituting in $s = \frac{1}{2}\ at^2$ the value of t given by $v = at$. This gives $s = \frac{1}{2}\ \frac{v^2}{a}$ or $v^2 = 2as$. These three formulæ, $v = at$, $s = \frac{1}{2}\ at^2$, and $v^2 = 2as$ are usually grouped together as the "three formulæ of uniformly accelerated motion." When applied especially to falling bodies, the symbol g is generally used instead of a (g representing the constant acceleration due to gravity). We have in that case $v = gt$, $s = \frac{1}{2}\ gt^2$ and $v^2 = 2gs$, the three formulæ for freely falling bodies starting to fall from a state of rest.

As has been stated previously in another connection, g, the change in velocity of a freely falling body in one

second, has been found by experiment to equal 980 cm. per sec. in each second in metric units, or 32.16 feet per second in each second in English units. These constants are usually written merely in the interest of brevity, 980 cm. per sec. per sec. or 980 cm. per sec.^{2} and 32.16 ft. per sec. per sec. or 32.16 feet per sec^2.

13. Equilibrium.—We will conclude this chapter with a brief study of the so-called *States of Equilibrium* of solid bodies. A cone balanced on its point is said to be in *unstable equilibrium*—the slightest touch will overturn it. The same cone resting on its base is said to be in *stable equilibrium*—if disturbed it returns to its original position. If the cone is lying on its side, it is said to be in *neutral equilibrium*—wherever it may be rolled there it will remain with no tendency to go further or to return. The explanation of these states is based upon a knowledge of the position of the *center of gravity* of the body. We can regard any body as being made up of a great number of small parts. The total pull of the earth on the body is then regarded as the resultant of the pulls of the earth on all the little component parts of the body. If we can find the point of application of this resultant pull on the body and then apply an equal upward force, this force will neutralize all of the little component forces and the body will remain stationary and balanced by the single upward force. With any given body we can in fact always find such a point by suitable experiments. This point which we can define as the point of application of the resultant of all of the parallel gravitational forces acting on the parts of the body, we call the *center of gravity* of the body. With symmetrical bodies the position of the center of gravity can be told by inspection—in a square or rectangle for instance it lies

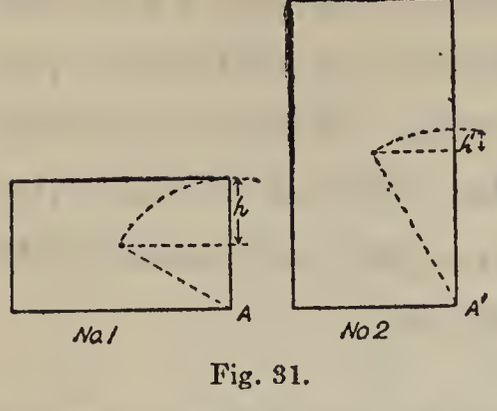

Fig. 31.

at the intersection of the diagonals, in a circle it is at the center, in a cube or sphere it is also at the center and so on.

Now the stability of a body depends on the work which must be done to overturn it—the amount of this work depends on the height thru which the center of gravity must be lifted in order to upset the body. Consider the two bodies shown in Fig. 31. In order to overturn body No. 1 around the corner A, the center of gravity must be swung along the arc shown and lifted thru the height h. In order to overturn body No. 2 around the corner A^1, the center of gravity need only be lifted thru the less height h^1. If these bodies are of equal weight, the stability of No. 2 is seen to be much less than that of No. 1.

We can now use these ideas to explain the States of Equilibrium. If any slight displacement of a body *raises* its center of gravity, then the body is in stable equilibrium, if a slight displacement *lowers* the center of gravity then the equilibrium is unstable. Finally if a slight displacement neither raises nor lowers the center of gravity, the equilibrium is neutral. We can readily apply these statements to the case of the cone resting successively on its base, point and side.

REVIEW.

1. What is (a) the equilibrant of a force? (b) the resultant of a number of forces?

2. Define (a) inertia, (b) momentum, (c) reaction.

3. What is meant by the statement that the earth's pull on a body varies directly with the mass of the body?

4. Distinguish between stable and unstable states of equilibrium.

CHAPTER V

WORK, ENERGY AND MACHINES

1. Work.— In the present chapter we shall consider the conditions under which forces can do work and learn how to measure the amount of work done as well as the rate of doing work in different kinds of simple machines.

In every action that involves movement a certain amount of work is done. In driving a nail or pushing a sled, a sufficient force must be applied to overcome whatever resistance may exist—the cohesion of the wood in the case of the nail and the friction of the runners in the case of the sled—and this force must act over a certain distance. Under such conditions work is done. The mere action of a force, however, without any motion being produced will never do any work. For instance, a column in a building may sustain a weight of many tons during a period of years, yet no work is done by the column. From the point of view of physics, work is done only when a force actually produces motion. The work done in any instance is measured by the product of the force acting multiplied into the distance over which it acts. If the force is represented by f and the distance acted over by s then the work done (W) is given by the expression $W = fs$.

The fundamental unit of force is, as was shown in the preceding chapter, the *dyne,* this being the force which if applied steadily to a gram mass will change its velocity one centimeter per second in each second. If this force acts over a distance of one metric space unit—that

is over one centimeter—the work done is one metric unit of work called a dyne-centimeter or an *erg*.

The dyne, being only $\frac{1}{980}$ of a gram force, is a very small unit—consequently the erg is also a small unit—too small in fact for practical use. In practice, physicists more commonly use as a unit ten million ergs. This larger unit is called a *joule*. Engineers however usually employ the pound force or the gram force rather than the dyne, so that the engineering unit of work is the pound force acting thru one foot which is known as the *foot-pound* or else the kilogram acting thru one meter which is known as the *kilogrammeter*. Of course these engineering units change slightly as we go from place to place on the earth's surface, since the values of the pound force and of the gram force are slightly variable.

If then in any instance, we wish to calculate the amount of work done by a force we merely multiply the value of the force in dynes, grams force, or pounds force into the distance in centimeters or feet thru which the force acts. The work done in raising 1,000 pounds one foot is 1,000 foot pounds; the work done in raising 1 pound 1,000 feet is also 1,000 foot pounds, as is the work done in raising 100 pounds thru 10 feet, and so on for other combinations. If the product fs is the same, the amount of work done is the same.

We must note especially that, possibly contrary to our preconceived ideas of work, the amount of work done in any operation has nothing whatever to do with the time taken to do the work. If a safe weighing two tons is lifted 30 feet to the second story of a building, the work done is fs or $4{,}000 \times 30$ foot-pounds, whether the safe is lifted in one second, one hour or one year by

a steadily applied force. It would take more *power* to lift the safe in one second than it would to lift it in one hour, but the total amount of work done would be the same.

2. Power.—This now brings us to the idea of *power*. The power used in any operation is measured by the amount of work done in a unit of time. Power is therefore defined in scientific terms as the time-rate of doing work. If we divide the total work done by the time taken to do it, we get the power used. Representing power by P we have $P = \frac{W}{t} = \frac{fs}{t} = fs$. If the safe were lifted thru the 30 feet in one second, the power expended would be $\frac{4000 \times 30}{1}$ foot-pounds per second. If it were lifted in one hour the power expended would be $\frac{4000 \times 30}{60 \times 60}$ foot-pounds per second. It is useful to have special units of power larger than the foot-pound per second or the erg per second. James Watt (1736-1819) the inventor of the steam engine, made certain experiments from which he concluded that an average horse could do work at the rate of 33,000 foot-pounds per minute or 550 foot-pounds per second. He called this unit the horse-power. This unit, frequently abbreviated H. P., is still in general use. To determine the H. P. used in any operation, divide the foot-pounds done per second by 550. The H. P. used to raise our safe in one second is seen to be $\frac{4000 \times 30}{1 \times 550} = 21.8$ H. P., while the H. P. used to raise it in one hour comes out $\frac{4000 \times 30}{60 \times 60 \times 550} = .06$ H. P. In the metric system the unit of power used in practice is not the erg per second, which is too small for convenience, but ten million ergs per second, that is, one joule per second. This unit is called a *watt*. It is well to remember that a H. P. equals

746 watts. In rating the power of large electrical machines, the *kilowatt* is used. This is, as the prefix indicates, one thousand watts equal to about one and one third H. P.

We shall now apply these ideas in a few simple problems:

(1) A hod carrier weighing 150 pounds carries a hod and mortar weighing 75 pounds up a ladder 25 feet high once every 20 minutes. How much work does he do in eight hours?

The total weight carried up the ladder each time is 150 pounds plus 75 pounds or 225 pounds. The height of the ladder is 25 feet. The work done on each trip ($W = fs$) is seen to be $225 \times 25 = 5625$ foot-pounds. This work is done three times an hour or 24 times in the eight hours. The total work done is therefore $2625 \times 24 = 135{,}000$ foot-pounds.

(2) How long will it take a 3 H. P. engine to lift 5000 bushels of wheat 50 feet? (A bushel of wheat weighs 60 pounds.)

A 3 H. P. engine can do 3×550 foot-pounds of work per second. The total number of foot-pounds of work to be done is $5000 \times 60 \times 50$ foot-pounds. The number of seconds which the engine will need in order to do this work will be $\frac{5000 \times 60 \times 50}{3 \times 550}$ or 9090.9 seconds or 150 + minutes. This result can be obtained of course by substituting directly in the formula $P = \frac{W}{t}$ or $t = \frac{W}{P}$.

3. Energy: Potential and Kinetic.— A very important principle of physics comes out at this point by observing that whenever work is done on a body, that body in turn becomes endowed with the ability to do work. For instance, after the laborer mentioned in the last paragraph

had carried his hod to the top of the ladder, the hod itself, if attached to a rope passing over a pulley and allowed to fall, could elevate a somewhat lesser weight at the other end of the rope and thus do work. Any body lifted to a height can, if allowed to fall back to its original level, do an amount of work roughly equal to the work done on it in lifting it.

So with a cannon ball fired from a cannon. A certain amount of work is done on the ball in setting it in motion. This work is stored up in the ball. When the ball strikes a target, it will do on the target an amount of work—destructive in this case—roughly equal to the work done on the ball in setting it in motion. So when a spring is wound up or compressed, and so in fact in all cases where force acting on a body moves or distorts it, that is to say, does work on it. The work stored up in a body which determines the ability of that body to do work is called the *energy* of the body.

The examples mentioned in the foregoing illustrate two different types of energy. The elevated weight possesses energy because of its *position,* while the moving cannon ball possesses energy because of its *motion.* The first type of energy, energy of position, is called *potential energy;* the second type, energy of motion, is called *kinetic energy.* It is important to note that the amount of energy of either kind possessed by a body depends upon the work done on the body in giving it its elevation or its motion. The energy of a body, being stored-up work, is measured in the same units that work is measured in. It is clear that the energy possessed by a body can never be greater than the work done on the body in giving it its energy. If there are no losses of energy such as those due to friction, which convert some

of the applied work into useless heat, the energy of the body ought to equal exactly the work done on it in all cases whether the energy is potential or kinetic in form. In practice, however, friction always enters into operations, wherefore, as we said above, the work done by the hod in falling would only roughly equal the work done in elevating it; this on account of the friction in the pulley. So with the cannon ball: the air resistance uses up some of the energy before the ball strikes the target, so that the work done on the target is not exactly equal to that done on the ball in giving it its motion.

In order to get these ideas definitely expressed, let us suppose that a ball of m pounds mass is lifted to the top of a tower s feet high. The work done on the ball will be m pounds force (the earth's pull on m pounds mass which must be overcome in lifting the ball) acting thru s feet, or ms foot-pounds. The potential energy of the ball—its stored-up work—when at the top of the tower will be exactly equal to the work done on it in raising it. Putting *P. E.* for potential energy, we have therefore $P.\ E. = ms$. If now the ball is allowed to fall to the ground, it will gain velocity as it falls and therefore gain kinetic energy, but at the same time lose potential energy on account of its lessened elevation. When halfway down, its energy will be half potential and half kinetic and when it strikes the ground its potential energy will be zero—all its energy will be kinetic. If we neglect the friction of the air, the kinetic energy of the ball at the instant it strikes the ground will be exactly equal to its potential energy when it was at the top of the tower and exactly equal to the work originally done on the ball in lifting it to the top of the tower. If we write *K. E.* for kinetic energy, we have therefore $K.\ E. = P.\ E. = ms$.

Since kinetic energy has to do with motion, it will be much more convenient to have a formula for it which involves the velocity with which the body is moving when it hits the ground. We get this velocity from the formula given in the preceding chapter: $v^2 = 2gs$ where g is the constant acceleration due to gravitation. From this $s = \frac{v^2}{2g}$ which substituted in $K.\ E. = ms$ gives $K.\ E. = \frac{1}{2}\frac{m}{g}v^2$. This formula enables us to calculate the kinetic energy in foot-pounds of a body of mass m pounds moving with a velocity of v feet per second. In working in English units g always equals 32.16 feet per sec.^{2}

For an example, find the kinetic energy of a ball with a mass of 200 pounds moving at the rate of 20 feet per second. $K.\ E.$ in foot-pounds $= \frac{1}{2}\frac{200}{32.16} \times (20)^2$. Working in metric units with $g = 980$ cm. per sec.^{2}, if the mass is expressed in grams, and the velocity in centimeters per second this formula gives the $K.\ E.$ in gram-centimeters. If the answer is wanted in dyne-centimeters that is in ergs, the value in gram-centimeters must be multiplied by 980, since 980 dynes equal 1 gram force. This is equivalent to writing $g \times K.\ E. = \frac{1}{2}\frac{m}{g}v^2$ for the original equation, whence it is seen that $K.\ E. = \frac{1}{2}mv^2$ will give the answer directly in ergs if we substitute the mass in grams and the velocity in centimeters per second.

In applying these formulæ for potential and kinetic energy in the solution of problems we should take great care to keep the various units properly related in accordance with the ideas brought out in the last paragraph. If in a problem the weight of the body is expressed in pounds or grams, this same number can be

used to represent the mass and when substituted for m in $K.\ E. = \frac{1}{2}\ \frac{m}{g}\ v^2$ will give the $K.\ E.$ in foot-pounds or gram centimeters, but if substituted for m in $K.\ E. = \frac{1}{2} mv^2$ the answers for $K.\ E.$ will be in foot poundals or dyne-centimeters (ergs).

Our discussion of the energy transformations in lifting the ball to the top of the tower and then dropping it, puts us in a position to understand better the fundamental theorem of the conservation of energy which we first stated in the Introduction. This theorem says that energy can be transformed from one state to another but can neither be created nor destroyed, the sum total of the energy in the universe being a constant. This theorem together with that of the conservation of matter forms the corner stone of the entire structure of the physical sciences. The energy expended in lifting the ball to the top of the tower was exactly equal to the potential energy of the ball after it was raised; in falling, this potential energy was converted exactly into an equal amount of kinetic energy possessed by the ball at the instant it struck the ground; and on striking the ground this energy was again transformed into energy of molecular motion, both the ground and the ball being slightly warmed.

In this last form, altho still a kind of kinetic energy, the energy is no longer available for doing external work and so apparently—altho only apparently—disappears. In any mechanical operation whatever if we carefully sum up the energy absorbed or put into the contrivance and the energy liberated or put out by the contrivance in the same time, we shall always find these two sums exactly equal. Of course in practice a large part of the output may be in the form of heat energy in the bear-

ings, axles and similar parts of the contrivance, or it may be frittered away in overcoming air resistance or in some other way so that at a first glance the output energy may seem much less than the input energy.

4. The Six Simple Machines.—A machine is a mechanical device for transforming or transferring energy. Machines are frequently defined as devices for doing work advantageously. All machines of however complicated a construction can be analyzed into a combination of one or more of the so-called "six simple machines." These are (1) the lever, (2) the pulley, (3) the inclined plane, (4) the wheel and axle, (5) the screw and (6) the wedge. As the wedge is simply a pair of inclined planes placed base to base, these six machines are readily reduced to five. These five we can group into two classes: (a) the lever, the pulley and the wheel and axle, all of which operate on the same principle, and (b) the inclined plane and the screw, which are essentially similar to each other.

These simple machines, and all others, operate under the one general law stated in the last paragraph but one: the energy absorbed in the mechanism must equal the energy liberated in it in the same time. Some of the liberated energy will be available to do external work; the rest of it will be used up in overcoming the various frictional resistances—those of the bearings, that of the air and so on. This can be stated in the following form: *The work put into a machine is equal to the work done on the machine to make it go—that is to overcome the friction—plus the work obtained from the machine and available to do external work.*

The work put into the machine can be represented by the products Fs—the applied force multiplied by the

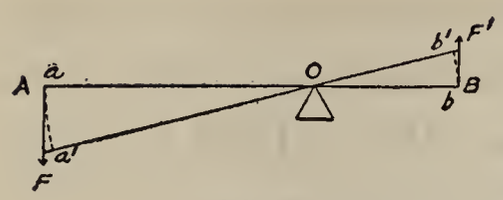

Fig. 32.

distance it moves—the work obtained available to do external work by F^1s^1—the force delivered multiplied by the distance thru which it moves. If the energy used up in friction is represented by R then the general law of machines can be stated $Fs = F^1s^1 + R$.

If we consider an ideal or perfect machine to be one in which the frictional resistance is zero, then for ideal machines $Fs = F^1s^1$. It is, of course, impossible to make such an ideal machine. Consideration of this law shows why it is impossible to make a "perpetual motion machine"—that is one which will go on forever without being continuously supplied with new energy. It is clear that the best that an ideal machine can do with all friction eliminated is to deliver an amount of energy equal to that put into it and that an actual machine must, on account of frictional losses, always deliver less energy than is supplied to it.

Let us now apply this general law of machines to the five simple machines, neglecting any consideration of frictional losses.

5. The Lever.—A lever is merely a rigid bar arranged to rotate around some fixed point. This fixed point is called the "fulcrum" of the lever. In Fig. 32 let the fulcrum of the lever AB be at o. Suppose a force F to be exerted at A as shown, since o is fixed a force say F^1 will be developed at B. If F acts thru the distance S represented by the arc aa^1, then F^1 will act thru the distance S^1 represented by the arc bb^1. Since the vertical angles at o are equal, the arcs aa^1 and bb^1 are proportional to their radii OA and OB, that is to say $\frac{aa^1}{bb^1} = \frac{OA}{OB.}$ Now by the general law of machines

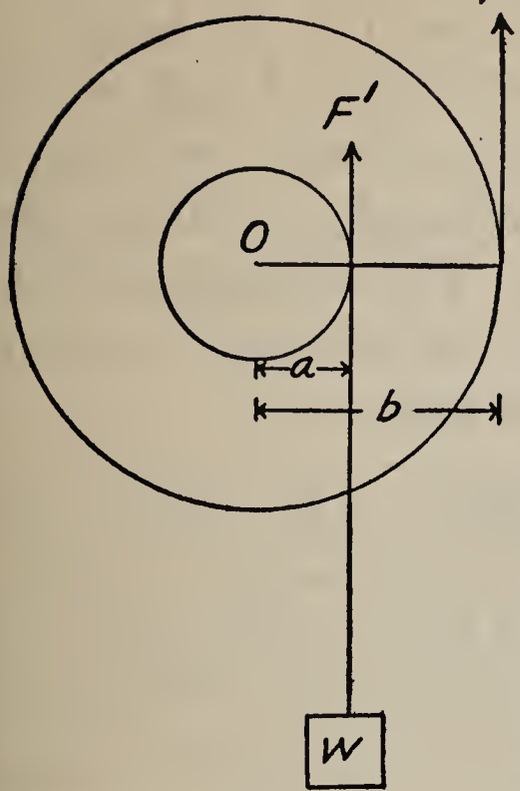

Fig. 33.

$F \times aa^1 = F^1 \times bb^1$ or $\frac{F}{F^1} = \frac{bb^1}{aa^1} = \frac{OB}{OA}$. OB can be called the resistance arm and OA the effort arm of the lever.

Using these terms, we can state as the special law of the lever that the force delivered bears the same relation to the force applied as the length of the effort arm bears to the length of the resistance arm. Accordingly, if the length of the effort arm is four times as great as the length of the resistance arm, then the force delivered will be four times as great as the force applied. This ratio of the force delivered to the force applied is known as the *mechanical advantage* of a machine. It will be noted that in the case of a lever the mechanical advantage is given directly by comparing the length of the effort arm with the length of the resistance arm.

6. The Wheel and Axle.— The lever is very widely used in the form of crowbars and similar appliances. It is clear that the travel of the force is very limited when using a lever since the end pushed on very soon comes flat on the ground. The principle of the lever is applied in the *wheel and axle,* when it is desired to make the resistance move over a considerable distance. Reference to Fig. 33 will show the relation of the pulley to the wheel and axle—the fulcrum is the shaft of the wheel at *o,* the effort arm is *b* and the resistance arm is *a.* The mechanical advantage of the device—that is the ratio of

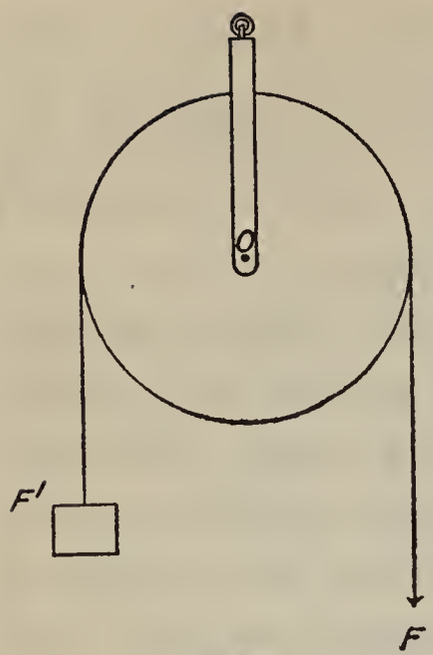

Fig. 34.

the force delivered to the force applied—is the ratio of the length of the effort arm to the length of the resistance arm—that is the ratio of the radius of the wheel to the radius of the axle. The windlass used to lift a bucket out of a well is a type of wheel and axle.

7. The Pulley.— The *single pulley* is merely an equal-armed lever. If *fixed* as in Fig. 34 its mechanical advantage is clearly *one*. The value of such a pulley is merely in changing the direction of application of the force acting—it is frequently more convenient to pull down than to pull up. The *single movable pulley* (Fig. 35) however, has a mechanical advantage of *two,* the fulcrum of the lever being at *o* and the effort arm *OF* being twice the resistance arm OF^1. It will be noted that since this pulley hangs in a loop of the rope, if the end of the rope at *F* is pulled up two feet, the pulley in the loop and therefore the load F^1 will be lifted only one foot. This consideration also shows the mechanical advantage to be *two,* for since by the general law $FS = F^1S^1$ then $\frac{F^1}{F} = \frac{S}{S^1}$, that is, the mechanical advantage can always be determined by comparing the distance moved by the applied force with the distance moved by the resisting force.

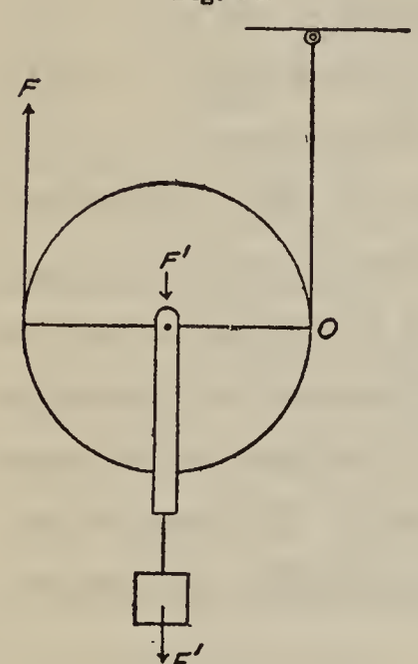

Fig. 35.

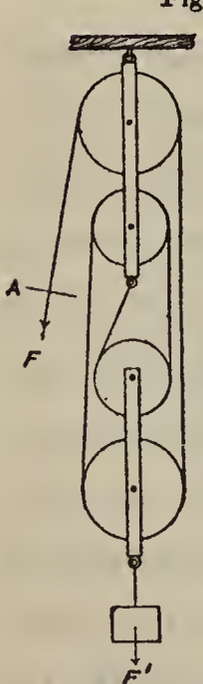

Fig. 36.

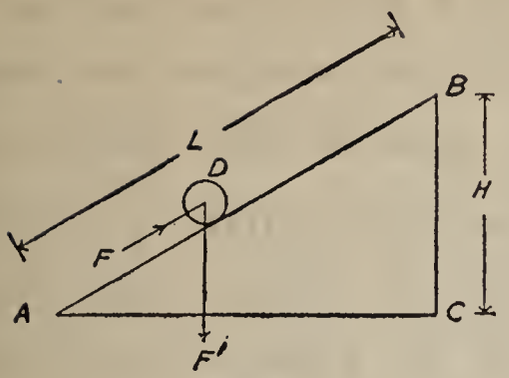

Fig. 37.

In order to determine the mechanical advantage of a *system of fixed and movable pulleys* such as is shown in Fig. 36 consider these facts: The lower block and the resisting weight are supported by four strands of rope. If one foot of rope is pulled past the point A this shortening of the system will be distributed among the four supporting strands so that the lower block will be lifted only $\frac{1}{4}$ of a foot; and so in general thru whatever distance the applied force moves, the resisting force moves only $\frac{1}{4}$ as far. This means that the mechanical advantage is four and that a given applied force at F will balance four times that force at F^1.

In any system of fixed and movable pulley, then, the mechanical advantage is equal to the number of strands of rope supporting the weight. This number of strands, it will be noted, is exclusive of the one to which the force is applied, for this strand does not directly support the weight.

8. The Inclined Plane.—We have already studied *the inclined plane* in connection with the composition and resolution of forces. It is very easy to get an expression for the mechanical advantage of this contrivance by a direct application of our general law of machines. Neglecting friction, it is seen that the work done by the applied force in moving the body D (Fig. 37) from the bottom of the plane to the top is Fl where l is the length of the plane. The work done against the resisting force, which is F^1, the weight of the body, is F^1h where h is the vertical height thru which the body is lifted. Now Fl

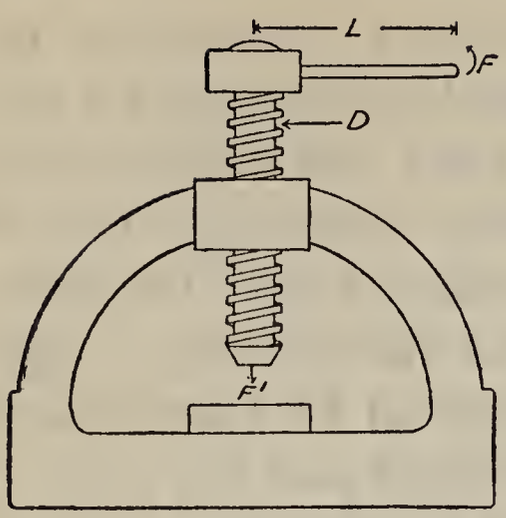

Fig. 38.

must equal F^1h, since we neglect friction; whence we see that $\frac{F^1}{F}$, that is the mechanical advantage equals $\frac{l}{h}$—the length of the plane over the height of the plane—provided that the line of action of the force is parallel to the surface of the plane.

9. The Screw.—The *screw* may properly be regarded as an inclined plane wrapped round a cylinder. The plane is the thread; the distance between two turns of the thread is called the *pitch* of the screw. When the screw turns in a nut, the screw advances a distance equal to the pitch for each revolution of the head. This fact enables us to determine the mechanical advantage of a screw, because when the applied force acts at the end of a lever of length l as in Fig. 38 it will travel a distance equal to $2\pi l$ while the screw advances a distance equal to the pitch d. The mechanical advantage is seen to be $\frac{2\pi l}{d}$ that is $\frac{F^1}{F} = \frac{2\pi l}{d}$.

We have now formula expressions for finding the mechanical advantages of each of the five types of simple machines. It will be noticed that where the resisting force to be overcome is greater than the applied force, the distance thru which the resisting force is pushed is smaller than the distance thru which the applied force must move, so that in all cases $FS = F^1S^1$—the work put in equals the work given out, friction being neglected.

10. Friction.—But in actual practice friction plays so large a part that it can never be neglected. It follows that the work output of a machine is always consider-

ably less than the work input. In practice the ratio of these two quantities—the work put in and the work delivered—is called the *efficiency* of the machine. A perfect machine would have an efficiency of unity or of 100%. Actual machines have efficiencies of all values from a few per cent up to 90% or thereabouts. As an example, systems of pulleys frequently show an efficiency of about 40%. The efficiency of screws is very low and of simple levers high.

The friction which causes these losses is due to the resistance which bodies offer to being pushed over one another. This resistance is largely due to small irregularities in the two opposed surfaces. It is much reduced by the use of an oil film which holds the two surfaces apart so that they do not actually touch. Altho in machines friction involves a loss of energy, yet in very many applications friction is distinctly advantageous. For instance, without friction, nails would not remain in wood, automobiles could not propel themselves along roads, men could not walk from place to place.

We know that friction between two surfaces develops heat. This is because the rubbing imparts extra motion to the molecules, making them move faster. As we have already learned, the latter process means that the temperature must rise. The physicist Joule (1818-1889) first showed that when one unit of mechanical energy is dissipated in friction, a certain definite fixed equivalent of heat is produced. The significance of these facts will be shown in the next chapter which will deal with the general subject of *Heat*.

REVIEW.

1. Define erg, joule, kilogrammeter.
2. Distinguish between potential and kinetic energy.
3. Is perpetual motion impossible? Why?
4. What are the five types of simple machines?

CHAPTER VI

HEAT

1. Heat and Energy.—In this chapter we shall discuss heat from the point of view of physics. Already in our work we have met with one or two scientific ideas about heat which will serve as an introduction to the subject. It will be remembered that in the chapter on the Molecular Theory it was said that the application of heat to a body causes the molecules of the body to move faster and that this results in a rise in the temperature of the body. Again in the chapter on Machines, it was shown that friction produces heat. It was also shown that this development of heat was accompanied by a certain apparent loss of energy.

In the light of the Theorem of the Conservation of Energy, according to which no energy can ever be lost, it seems logical to conclude that the heat developed in the friction in a machine is a form of energy. It would seem further that the development of this energy must be dependent on the rate of motion of the molecules of the body. We might therefore surmise that the heat of a body is connected with the energy of motion of its molecules, or, in general, that heat is a form of energy, namely, the *kinetic energy of molecular motion.* All the experimental evidence of physics points toward this same conclusion, so that we may in fact say with complete confidence that *heat is the kinetic energy of molecular motion.*

This view of the nature of heat was first suggested

about 1798, by Benjamin Thompson, Count Rumford, a very able and interesting native of Massachusetts. Rumford performed some fairly conclusive experiments covering this point, but the matter was finally settled by James Prescott Joule in work, the first of which was published in 1842. Prior to the introduction of this Mechanical Theory of Heat, heat was supposed to be a weightless fluid called "caloric." A hot body was supposed to have more "caloric" than a colder one and the fluid was supposed to flow from the hotter to the colder body. This theory of "caloric" will indeed explain most of the phenomena of heat excepting those connected with the development of heat by friction.

2. Temperature.—We all have more or less clear ideas of what is meant by "temperature." When a body feels hot we say it has a high temperature; when it feels cold we say it has a low temperature. We know that if we put an iron in a flame and thus impart heat to it, it will come to a higher temperature. Temperature, then, is a word which we use to denote the degree of hotness or coldness of a body. It is important to notice that "cold" is merely the absence of heat. The more heat we remove from a body the colder it gets and the colder it feels.

The judgments of temperature which we form by the sense of feeling with the hand alone are very deceptive—depending both on the immediately previous heat treatment of the hand and on the rate at which the body felt of carries heat away from the hand. For example, if the hand has just been dipped into hot water, tepid water will feel cold. Also a piece of cold iron which allows the heat to escape from the hand very rapidly

Fig. 39.

feels much colder than a piece of wood of the same real temperature. As it is of great practical importance to be able to compare the temperatures of different bodies, it is clearly very desirable to have some accurate device for making these comparisons. Such devices, of which there are a great variety, are called " thermometers."

3. The Thermometer.—Galileo, in 1592, devised the first thermometer. Its construction is shown in Fig. 39. Galileo had observed the fact which we have noted in Chapter III, that the pressure exerted by a gas changes with its temperature, altho he did not know the true cause of this change. When a hot body is brought in contact with the air bulb at the top of the device in Fig. 39, the pressure of the air will increase and the surface of the liquid in the narrow tube will be forced down a certain distance until the pressure inside equals the pressure outside, minus that caused by the head of liquid in the tube. The hotter the body, the lower the liquid will be forced. Two bodies which force the liquid down equal amounts are taken to be equally hot, that is to have the same temperature.

Liquids—and solids also—expand when heated for the same reason that gases expand, that is, because of the increased rates of motion of the molecules. The air thermometer of Galileo was awkward to handle and had a small range unless the tube was made very long. About the year 1700, the more convenient mercury thermometer was invented to replace the air thermometer. Nearly all present-day thermometers are of glass containing either mercury or alco-

hol, the latter liquid, when used, being colored red or blue.

In order to understand exactly what the scale marked on one of these thermometers represents, it is best to study the mode of manufacturing a mercury thermometer. First, a piece of " thermometer tubing " is taken and a bulb is blown at one end. The bore of this tubing, that is the internal diameter of it, is very small—about the diameter of the very finest sewing needles. The front of the tube is made in such a shape that it acts as a magnifying glass and makes the liquid column, when viewed from in front, look much wider than it really is. The bulb is filled with mercury and the whole heated to a temperature higher than the highest temperature at which the thermometer is subsequently to be used. At this temperature, the tube is sealed at the top where the mercury is running out. If the thermometer at any subsequent time is heated to this same temperature it will, of course, burst.

The next step is to calibrate the thermometer, that is, to put the scale on it. After the mercury has shrunk back into the tube, leaving a vacuum above it, the bulb is put into a vessel containing melting ice. Now it has been found that the temperature of melting pure ice under ordinary pressure is always the same, so that it can and does serve as a *fixed point* in temperature. All thermometers no matter where manufactured have the melting point of ice as one fixed point on their scales. After the mercury has become stationary above the ice, a scratch is made on the glass to show the exact level. The thermometer is then taken out and next immersed in steam from boiling water in such a way that the bulb and stem are both surrounded with steam which is es-

caping freely into the air and is therefore under atmospheric pressure. If the atmospheric pressure is normal, that is 76 centimeters, the temperature of free steam from boiling water is always the same, so that it serves as a *second fixed point* in temperature. All thermometers have this second fixed point marked on their scales. The final level of the mercury is indicated by a scratch on the glass as before. The thermometer now has two definitely fixed points. If at any subsequent time the mercury stands at the lower scratch, then the temperature of the bulb must be the same as that of melting ice—if it stands at the upper scratch, the temperature of the bulb must be the same as that of free steam under normal pressure. It is assumed that under the influence of rising temperature, the mercury in the bulb will expand uniformly from the one fixed point to the other, so that to compare intermediate temperatures, it is only necessary to assign numerical values to the fixed points, and then to divide the intermediate space into a convenient number of equal parts.

4. Centigrade and Fahrenheit.—There are in common use two different systems of subdividing the thermometer scale. The first of these, the scientific or Centigrade scale, was devised by Celsius of Upsala, Sweden, in 1742, and is frequently called the Celsius scale. In that scale the lower fixed point is marked 0 and the upper fixed point 100. There are 100 intermediate divisions called degrees Centigrade. A temperature of 40 degrees Centigrade is written 40° C. This scale is in universal use for scientific purposes, thruout the world and for domestic purposes in most parts of western Europe. The second scale, which is the one in common household use thruout the United States and England, is the

Fahrenheit scale. This was devised by Fahrenheit of Danzig, Germany, in 1714. The lower fixed point is marked 32 and the upper fixed point 212. There are 180 equal intermediate divisions called degrees Fahrenheit. Forty degrees on this scale is written 40° F.

It is important to be able to transform readings on one of these scales into the corresponding readings on the other. It is easy to see how this can be done. Since the fixed points on all thermometers are the same, we have 100 Centigrade degrees representing the same change in temperature as 180 Fahrenheit degrees. One degree Centigrade is therefore clearly equal to $\frac{180}{100}$ of a degree Fahrenheit, that is $\frac{9}{5}$° F., and conversely one degree Fahrenheit must equal $\frac{5}{9}$° C. Now, if we wish to convert a reading of 40° C. into Fahrenheit—we note that the reading means 40 Centigrade degrees above the lower fixed point which is marked 0 in this scale. This corresponds to $40 \times \frac{9}{5} = 72°$ F. above the lower fixed point. But on the Fahrenheit scale the lower fixed point is marked 32°, hence the actual reading on the Fahrenheit scale corresponding to 40° C. will be 72° plus 32° = 104° F. If we represent the Centigrade reading by C and the Fahrenheit reading by F this result can be expressed in the following formula: $F = \frac{9}{5} C + 32$. If now, we wish to convert 40° F. into the corresponding Centigrade reading, we must first find how many Fahrenheit degrees above the lower fixed point this reading represents. Since the lower fixed point on this scale is marked 32°, it is obvious that $40 - 32 = 8$ is the number we seek. 8° F. above the lower fixed point equals $8 \times \frac{5}{9}$ = 4.4° C. above the same point. As this point on the Centigrade scale is 0°, the final Centigrade reading cor-

responding to 40° F. is seen to be 4.4° C. The formula expression for this calculation is $C = (F - 32)\ \frac{5}{9}$.

5. Measuring High and Low Temperatures.— The thermometer filled with mercury is, speaking generally, more reliable than any other thermometer containing liquid because of the great uniformity of the expansion of mercury over ordinary ranges of temperature. Mercury however freezes at 39° C. below Centigrade zero and boils at 360° C. For temperatures below —39° C. thermometers containing alcohol can be used since the freezing point of alcohol is —130° C. For temperatures above 360° C., the mercury thermometer can still be used if the space over the mercury is filled with nitrogen gas. The pressure of this gas prevents the mercury from boiling, so that such thermometers can be used up to 700° C., which is nearly that of red hot iron. In ordinary mercury thermometers, the space over the mercury is a vacuum except for the presence of mercury vapor.

For the measurement of the highest temperatures, such as those in furnaces and pottery kilns, electrical devices are used or certain optical arrangements in which the temperature is measured by the brightness of the interior of the furnace. Such high temperature measurers are called *pyrometers.* The highest known temperature is that of the sun, estimated at 5700° C. The highest terrestrial temperature is that of the electric arc, about 3500° C.

In scientific work when very accurate temperature measurements are wanted over either high, low or medium ranges, the standard gas thermometer is always used. In this instrument (Fig. 40) the bulb *A* contains hydrogen gas. Mercury is introduced into the pipe as

Fig. 40.

shown by the black line and the space over the mercury in the closed tube *B* is made a vacuum. The pipe contains a flexible rubber link at the bottom. The bulb *A* is put in melting ice and the pipe *B* moved up or down until the mercury level is brought to a fixed mark *a*. The pressure of the hydrogen at this temperature is found to balance the mercury column *a* to *b*. *b* is then marked as the "lower fixed point." When *A* is in steam, the pressure after the level of mercury on the left is again brought to *a* will be found to balance a longer column say *ac*. This makes *c* the "upper fixed point." The scale is usually divided into Centigrade degrees, the lower fixed point being marked 0° and the upper fixed point 100°.

In reading a temperature with this thermometer, tube *B* is always to be moved until the lower mercury level stands at *a* before a reading is made. The most evident advantage of this hydrogen thermometer lies in its great accuracy at very low temperatures. Extreme low temperatures are nearly always measured by its use.

The lowest temperatures which have been measured are listed here. In 1845 Faraday evaporated a mixture of ether and solid carbon dioxide in a vacuum and attained a temperature of —110° C. In 1880, when air was first liquefied, it was found to have a temperature of —180° C. In 1900 Dewar, by evaporating liquid hydrogen, reached —260° C., and finally in 1911 Kammerlingh Onnes, of Leyden, Holland, by evaporating liquid helium obtained a temperature of —271.3° C. There is strong reason to believe, as we shall see later, that the lowest possible temperature is —273° C. This tempera-

ture is known as the "absolute zero." It is the temperature at which the molecules of matter cease to move, so that they have no kinetic energy and therefore the body has no heat to lose and cannot get colder. This matter is discussed at greater length later in this chapter.

6. Transference of Heat: Conduction.—We all know that a hot body placed in contact with a colder body will warm the colder body. Let us now consider the various ways in which heat can be transferred from one body to another and from one place to another.

If we take a number of equally long wires of different metals, as silver, copper and iron and after twisting all the wires together at one end and spreading the other ends out fanlike, heat the twisted ends in a flame, we shall find that the far ends of the wires all become hot in time but that certain of the wires get hot at the far end much more quickly than others. The silver wire would become hot thruout the most quickly—the iron wire the least quickly. In this experiment heat is evidently transferred along the wire by direct molecular contact. The rapidly moving molecules at the hot end strike on the adjacent molecules and thus transmit their increased motion thruout the wire. This mode of transferring heat which is effective only from one part to another of the same body or from one body to another in contact with it, is called transference by *conduction.*

As our experiment showed, the rate at which heat is conducted is very different for different materials, that is to say, different materials have very different *conductivities* for heat. Accurate experiments show that if the conductivity of silver is rated as 100, that of copper

is 74, while that of iron is only 12, and that of glass only about .046. This means that for rods of equal cross areas and equal lengths, with equal differences of temperature between their ends, a silver rod will transfer 100 units of heat in the same time that an iron rod transfers 12 and a glass rod transfers .046 of one unit. Silver is the best conductor known. All liquids have low conductivities—that of water being about $\frac{1}{1200}$ of the conductivity of silver. Gases have still lower conductivities, the conductivity of air being about $\frac{1}{25}$ that of water.

If we wish to keep our bodies warm in cold winter air we must cover them with poor conductors of heat. Most of the warmest coverings, as fur, feathers, wool and so on, owe their low conductivity to the large amounts of air imprisoned in them, still air being one of the poorest known conductors. Snow and ice also have low conductivities (about .21 on the scale used above) so that a blanket of snow over the ground will protect the roots of trees and plants from freezing. Ice houses, such as are used by Eskimos, are also warm in that they retain the heat of fires built within. In our climate, houses for storing ice are made with double walls and the interspace filled with sawdust. The sawdust imprisons air and prevents its free circulation, so that a wall thus constructed keeps the summer heat away from the ice.

7. Convection.—The importance of preventing circulation of the air in the interspace of the icehouse wall can be better understood when we study the transference of heat by what is known as *convection.* Consider the air in a closed room with a hot stove at one side. The air

around the stove is heated and therefore expands, becoming less dense than the rest of the air in the room. The air around the stove must then rise, being pushed up by the denser air in accordance with Archimedes' principle. The cooler air from along the floor now comes in around the stove, is heated, expands and rises in its turn. We have thus a constantly ascending column of heated air over the stove. This strikes the ceiling, flows along it and down the other wall, cooling as it goes and finally comes back to the stove again across the floor. We have thus maintained in the room steady *convection currents* which transfer the heat from the stove into all parts of the room.

The heat from all forms of internal heating apparatus like stoves, hot water and steam radiators and hot air registers is distributed thru our rooms by convection. It will be noted that in this mode of heat transference, an entire mass of the heated material is transferred, containing the heat in it. Convection is therefore essentially different from conduction in which the molecular motions, the energy of which constitutes heat, are transferred from molecule to molecule thruout the body.

Convection is responsible for the production of winds and ocean currents. The air in the tropics becoming heated, is pushed up by colder air rushing in from higher latitudes and this, coupled with the rotation of the earth, causes the steady air movements known as the "trade winds." All winds, speaking generally, are caused by convection currents between areas of unequal heating. In the same way, most ocean currents result from the unequal heating of the ocean between the tropical and arctic zones. Convection is

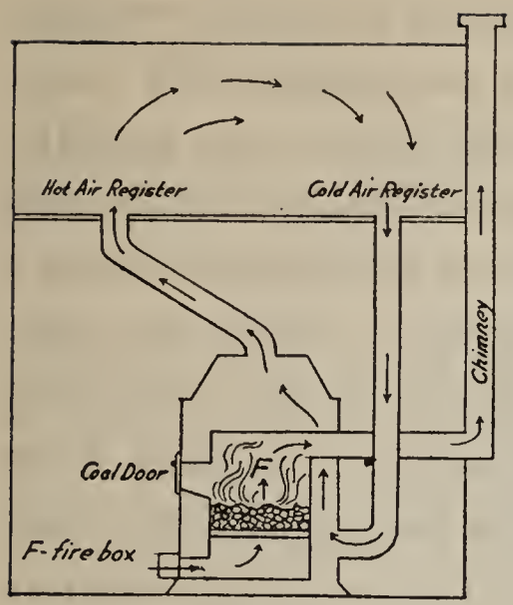

Fig. 41.

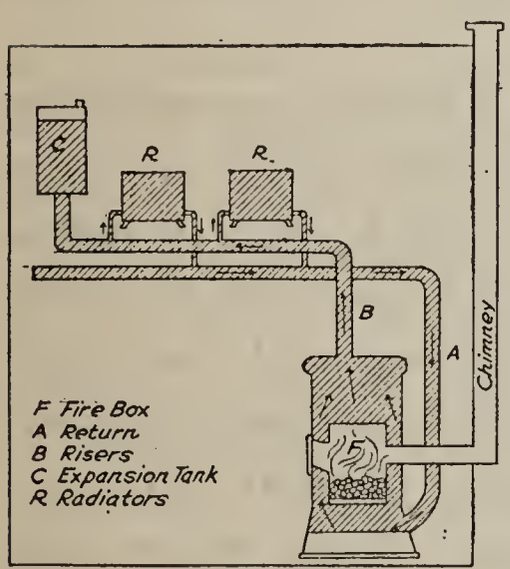

Fig. 42.

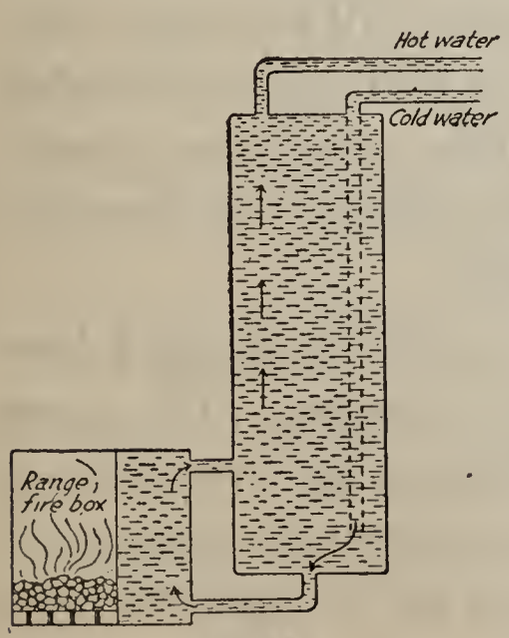

Fig. 43.

also responsible for the operation of household hot water boilers and similar appliances.

Fig. 41 represents a hot air heating system—Fig. 42 a hot water system and Fig. 43 a kitchen boiler. The arrow heads show the circulation of the fluid in the system. Taken in connection with the explanations of convection already given, the arrows serve to show exactly how each arrangement operates. In the hot air system shown the same air is used over and over again, which is not desirable. In the best hot air systems, fresh cold air is taken in from outdoors thru a "cold air box." This air after being heated and circulated in the upper rooms leaks out thru the ordinary ventilating channels and is continuously replaced with fresh heated air.

8. Radiation.—The third and last method of heat transference is by *radiation.* We shall be better able to understand radiation after we have studied the subject of light. We can nevertheless present a brief explanation here. All the universe is

supposed to be filled with a weightless or nearly weightless fluid called the *ether,* which surrounds and penetrates all matter. When molecules move very rapidly, they make a disturbance in this ether which travels outward from the body as a water wave travels out when a stone is thrown into a pond of water. When this disturbance results from the heat motion of the molecules, the resulting train of waves is known as *radiant heat.* When these waves strike some other body of matter they set the molecules of that body into more rapid motion, just as the water wave in the pond disturbs the pebbles on the shore when it arrives there. The second body will thereby be warmed.

This mode of transferring heat is known as *radiation.* It is of importance because all the heat which comes to us from the sun is carried by this method. It is of course clear that heat from the sun cannot come to us either by conduction or convection. It is interesting to note that the waves of radiant heat travel with enormous speeds—their velocity being approximately 186,000 miles per second—the same as that of light. These waves may pass thru a body without warming it. Many materials are very transparent to the waves. Ordinary window glass for instance is. The sun's heat will therefore pass freely thru the lids of glass hot beds and will be absorbed by the earth inside. The heat after absorption cannot get out thru the glass, whence the efficiency of these devices in forcing the growth of plants.

Again, radiant heat like light travels in straight lines and can therefore be shut off by a screen. The peculiarly burning heat which can be felt on the face from an open fire is radiant heat. It is a matter of common experience that this heat can be cut off completely by a

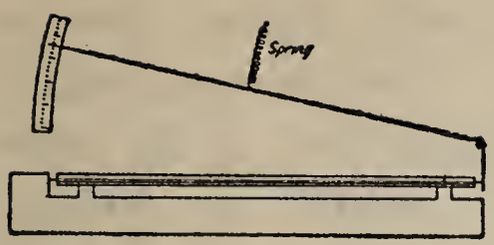

Fig. 44.

piece of paper or similar material held in front of the face.

9. Effects of Heat on Matter.—We have now completed our study of the modes of heat transference, which may be summed up by saying that heat is transferred by conduction, convection and radiation. We turn next to the *effects* produced in matter by heat. The most evident of these effects is *expansion.*

We have already spoken of the expansion produced by heat. On the basis of our theory of heat we would naturally expect expansion to result from heating, since the increased molecular motions will cause the molecules to strike harder blows on one another and thus drive themselves farther apart. The expansion of solids under the influence of heat can be measured by making the material up in the form of a rod and measuring its length first at the temperature of melting ice and then at the temperature of steam. This measurement can be made by a simple lever device such as is shown in Fig. 44. If we divide the increase in length by the number of degrees rise in temperature and by the original length, we shall get the increase in length per degree for each unit of length. This quantity is called the "coefficient of linear expansion" of the material. It is of importance because knowing the length of a bar of the material at any temperature and this coefficient, we can calculate its length at any other temperature.

For instance: What is the increase in length of a line of copper wire 10 kilometers long when the temperature rises from 0° C. to 25° C.? Experiments have shown the coefficient of linear expansion of copper to be .000017—that is the gain in length per centimeter

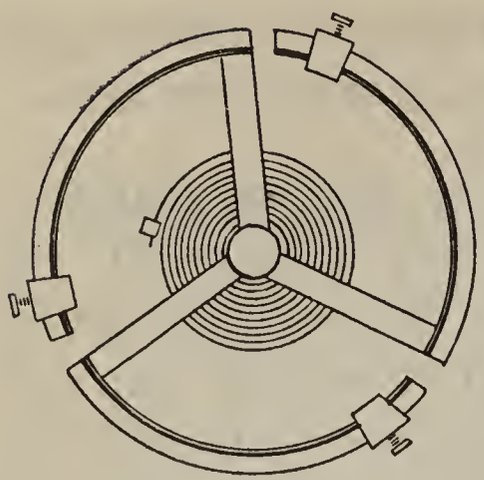

Fig. 45.

length for each degree rise is .000017 cm. The gain in length per unit length for a 25° rise will be $25 \times .000017$ or .000425 cm., and the gain in length in 10 kilometers which equal 10×1000 meters or $10 \times 1000 \times 100$ centimeters will be 425 centimeters.

10. Coefficients of Expansion—Solids.— Different substances have very different coefficients of expansion as can be seen from this list of coefficients:

Brass	.000018	Glass	.000009
Iron	.000012	Tin	.000022
Steel	.000011	Zinc	.000029

Numerous practical applications are made of this property of unequal expansion. If a bar made up by riveting together thin strips of, say, brass and steel is heated, the brass will expand more than the steel so that the bar will bend. The balance wheels of watches are compensated so that their periods of vibration will be independent of the temperature in a manner suggested by the bending of this compound bar. Increase in temperature increases the radius of an ordinary balance wheel which makes the wheel vibrate more slowly so that the watch loses time. The rim of a compensated wheel is made up of a compound strip cut into sections as shown in Fig. 45. The inner metal has the smaller coefficient of expansion, so that as the temperature rises, the ends of the sections are bent in and the effective radius of the wheel thus reduced just enough to compensate for the general expansion. The period is thus kept constant. It is of course evident that the wheel needs to

Fig. 46.

be carefully adjusted if it is to be entirely accurate.

Fig. 46 shows another application of unequal expansion, this time to make a compensated pendulum for a clock—that is to say a pendulum the length of which will not change with the temperature. An ordinary metallic pendulum becomes longer when it warms up so that the clock loses in hot weather. In this compensated pendulum the rods of one material are arranged so that their expansions let the pendulum down while the rods of the other material raise it. The lengths are so selected that the bob is not moved either up or down by the temperature change. The materials ordinarily used are steel and copper.

The thermostats which are used in houses to regulate the heating apparatus when the temperature goes too high or too low, are made on the same principle, the movement of the compound strip in this case closing an electrical circuit which produces the necessary alteration in the furnace draughts.

11. Expansion of Liquids.— The expansion of liquids is of considerable importance in connection with their use in thermometers. The coefficient of expansion of a liquid can be determined just as was that of a solid by heating the liquid in a narrow tube and then making the necessary allowance for the expansion of the tube. When this is done the coefficients for liquids are found to be greater than for solids. For example, the coefficient of alcohol is .001, of ether .0015 and of water .0002. These coefficients are *volume coefficients:* that is they represent the increase in volume of a unit volume for a

rise in temperature of one degree. It is evident that a linear coefficient would be of no use in dealing with liquids which do not permanently retain any particular shape.

Experiments show further that the expansion of liquids is irregular—that is to say, the coefficient of expansion is generally different if measured over different ranges of temperature except in the case of mercury which exhibits great regularity of expansion. It is on account of this regularity that mercury is so widely used in thermometers.

The change in volume of water with change in temperature is so peculiar that it deserves special attention. When the temperature of pure water is steadily reduced from 100° C., the water contracts until 4° C. is reached—after that as the temperature falls from 4° C. to 0° C. the water will be found to *expand.* It is evident from this that water has its greatest density at 4° C. This seemingly unimportant fact has important consequences in nature in affecting the way in which bodies of water freeze.

The water in a lake cools from the surface. As the surface water becomes cooled, it will sink to the bottom being replaced by warmer water from underneath. This circulation goes on until the temperature of all the water becomes 4° C. When the surface layers cool below this temperature they become less dense than before. There is consequently no tendency for them to sink and the surface layer remains in place until it freezes. The freezing of the water underneath then takes place from the under surface of the ice. If the water contracted continuously until it reached its freezing point the lake would begin to freeze from the bottom

instead of from the top and would finally freeze solid. All living things in the water would be killed and it is unlikely that any except the most shallow lakes would thaw completely in the summer.

12. Expansion of Gases.— Let us next consider the volume changes in gases under change in temperature. Since gases are so easily compressible, it is very necessary in studying their rates of expansion to arrange that the pressure acting on the gas experimented on shall at all times be kept the same. If this is done, then coefficients of volume expansion can be obtained for gases just as for liquids and solids. The volume coefficient of expansion of a gas is the change in volume of a unit of volume at 0° C., for one degree change in temperature, the pressure remaining fixed, just as the coefficient of linear expansion of a solid is the change in length of a unit length for one degree change in temperature.

The Frenchman, Gay-Lussac, in 1802, determined the coefficients of volume expansion of a large number of gases, using all necessary precautions to maintain a constant pressure. He made the discovery that *all gases have the same coefficient of volume expansion.* The value of this universal coefficient is .00367 or $\frac{1}{273}$. This is known as Gay-Lussac's Law.

Charles, another Frenchman, had already, in 1787, made use of the hydrogen thermometer (Fig. 40) to determine the pressure coefficients of a long series of gases. The pressure coefficient is defined as the change in pressure of a fixed volume of gas per unit of the pressure at 0° C. for each degree change in temperature. Charles found that *the pressure coefficients of all gases are the same.* The value of this coefficient is .00367 or

$\frac{1}{273}$. This is called Charles' Law. It will be noted that this law was discovered before the law of Gay-Lussac. Knowing Boyle's law (that at constant temperature the volume of a mass of gas is inversely proportional to the pressure) Gay-Lussac's law can be deduced from the law of Charles. Consequently these two laws are frequently stated as one. The statement is then called the Law of Charles and Gay-Lussac.

13. Law of Charles and Gay-Lussac.— Let us consider somewhat in detail the meaning of these laws. Charles' Law says that the pressure coefficient of all gases is $\frac{1}{273}$, which, as we have already implied, means that if the volume is fixed, then for every degree Centigrade rise in temperature, the pressure of an inclosed mass of gas increases $\frac{1}{273}$ of the pressure that it exerted at 0° C. If therefore we raise the temperature to + 273° C. the pressure will be doubled. On the other hand *if we lower the temperature to 273° C. below 0° C., then the pressure will be zero.* Since the pressure is zero, the molecular motions must have ceased entirely. If the molecular motions have ceased, then their kinetic energy must be zero and the gas can contain absolutely *no heat energy.* Again, since the gas has no heat energy, no heat can be taken from it and consequently it is impossible for the gas to get any colder. This temperature, —273° C. is therefore known as the *Absolute Zero*—the temperature at which bodies contain no heat. It will be remembered that Kammerlingh Onnes, in 1911, produced a temperature of —271.3° C. by evaporating liquid helium. No way is known to get a lower temperature than this so that the absolute zero has as yet never been reached.

Gay-Lussac's Law, interpreted in the same way, indicates that the volume of all gases at —273° C. is zero. This means really that the molecular motions having ceased, the intermolecular spaces will be reduced to zero —the total volume occupied by the gas being then merely the sum of the volumes of the solid molecules. It should be noted that in practice all known gases become liquids before the absolute zero is reached, and having become liquids, they no longer follow these two gas laws. It results therefore that the deductions made from these laws as to conditions at —273° C. can never be definitely tested.

14. The Absolute Scale of Temperature.— Since —273° C. is the temperature at which molecular motions cease and at which matter contains no heat, this temperature is clearly the proper zero point at which to start a temperature scale. The scale beginning at this zero is known as the *Absolute Scale of Temperature.* It is very widely used in scientific work. The degrees on this scale are Centigrade degrees so that 0° C. corresponds to 273° absolute, or 273° A., as it is written. In order to change any Centigrade reading to the absolute scale, it is only necessary therefore to add 273 to the Centigrade reading or, as a formula: $A = C + 273$.

The use of this scale of temperature makes it possible for us to restate the Laws of Charles and Gay-Lussac in more compact form. If the pressure exerted by a volume of gas is 273 units at 0° C. (that is 273° A.), it will be, by Charles' Law, 0 units at 0° A., 50 units at 50° A. and so on—*the pressure* in other words, if the volume remains fixed, being *directly proportional to the absolute temperature.* If the temperature expressed on the absolute scale is doubled, the pressure will be doubled and

so on. If we write T for the absolute temperature of the gas, then we can put $P \propto T$ *or* $\frac{P_1}{P_2} = \frac{T_1}{T_2}$ where P_1 and P_2 represent different pressures corresponding to the temperatures T_1 and T_2. This follows from the facts stated in the Introduction, because if $P \propto T$ then $\frac{P}{T} =$ a C is the temperature at which molecular motions cease constant number, say k; so $\frac{P_1}{T_1} = k$ and $\frac{P_2}{T_2} = k$ and so forth, whence we see that $\frac{P_1}{T_1} = \frac{P_2}{T_2}$ and this can be written $\frac{P_1}{P_2} = \frac{T_1}{T_2}$.

Following exactly similar reasoning, the law of Gay-Lussac can be stated: *the volume of a gas is directly proportional to the absolute temperature* if the pressure remains fixed. This statement expressed as a formula is $V \propto T$ or $\frac{V}{T} = k$ and $\frac{V_1}{T_1} = \frac{V_2}{T_2}$ or $\frac{V_1}{V_2} = \frac{T_1}{T_2}$.

We now have the two laws stated as two proportions: $\frac{P_1}{P_2} = \frac{T_1}{T_2}$ and $\frac{V_1}{V^2} = \frac{T_1}{T_2}$, the first of which is true for variations of P and T if V remains constant, while the second is true for variations of V and T if P remains constant. Now since both the pressure P, and the volume V of a gas are directly proportional to the absolute temperature T, their product PV will also be directly proportional to the absolute temperature. Wherefore we can write $PV \propto T$ or $\frac{PV}{T} =$ a constant number, say R; so $\frac{P_1V_1}{T_1} = R$ and $\frac{P_2V_2}{T_2} = R$ and so on; whence we can write $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ or $\frac{P_1V_1}{P_2V_2} = \frac{T_1}{T_2}$ in which P_1 V_1 and T_1 refer to the pressure, volume and temperature (absolute) before any change takes place and P_2 V_2 and T_2 refer to conditions after the change.

This last equation is usually referred to as the *General Gas Law.* It is of practical importance in that by the use of it we can solve any problem dealing with volume,

temperature and pressure changes in a given mass of gas. For example, suppose that at 20° C. and 740 mm. pressure, the volume of a mass of gas is 500 cubic centimeters. Find the volume that this same mass of gas would have at 0° C. and 760 mm. pressure.

$P_1 = 740$ $P_2 = 760$

$V_1 = 500$ $T_2 = 273° A$ since $\frac{P_1 V_1}{P_2 V_2} = \frac{T_1}{T_2}$

$T_1 = 273 + 20 = 293° A$ $V_2 = X$

we have $\frac{740 \times 500}{760 \times X} = \frac{293}{273}$ whence $273\ (740 \times 500) = 293\ (760 \times X)$ and $X = 453.6$ cm.^{3}

REVIEW.

1. State the mechanical theory of heat.
2. Convert 50° Centigrade into Fahrenheit.
3. Distinguish between conduction, convection and radiation.
4. What is meant by coefficient of expansion?
5. Why is it that a body of water freezes from the top and not uniformly thruout?

CHAPTER VII

QUANTITY OF HEAT, CHANGE OF STATE AND HEAT ENGINES

1. Quantity of Heat.—Having completed our study of the expansion effects produced in solids, liquids and gases by heat—we next turn to a consideration of changes in the *states* of matter—solid, liquid or gaseous—produced by heat. In order to discuss this matter intelligently, it will be necessary first to have some definite ideas on the subject of *quantity of heat.* So far we have been talking about temperature and the effects produced by changes in temperature without paying attention to the quantities or amounts of heat required to bring about the different changes.

We all know that with a fixed flame it takes much longer to raise a gallon of water to its boiling temperature than it does to raise a pint of water to the same temperature. If we consider heat as the kinetic energy of the moving molecules, it is evident that the gallon of water at 100° C. contains much more heat energy, that is to say, a much greater quantity of heat, than the pint of water at the same temperature. The former contains, in fact, eight times as much heat as the latter, since there are eight pints in a gallon. It is clear then that the amount of heat required to produce a certain temperature change in water depends on the mass of water heated. The amount also naturally depends on the number of degrees of temperature thru which the water is raised. Thus it takes $\frac{1}{5}$ as much heat to raise a mass of water thru 10° C. as it does to raise the same mass thru 50° C.

If now in place of the pint of water we put over the fixed flame a piece of copper of the same mass as the pint of water, we shall find that the copper will rise to 100° C. in just about $\frac{1}{10}$ of the time taken by the water. This brings out the fact that different substances require different amounts of heat in changing their temperatures over the same range even when equal masses are used.

In order to measure and compare the amounts of heat taken up by different bodies heated under different conditions, we must have a *unit of heat quantity.* It has been found convenient to take as the unit of heat quantity the amount of heat absorbed by one gram of water when its temperature is raised one degree Centigrade. This unit is called the *gram-calorie* or for short *the calorie.* The same amount of water when lowered one degree in temperature will of course give out this same quantity of heat so that the *gram-calorie* can be defined as the quantity of heat *taken in* or *given out* when the temperature of one gram of water is *raised* or *lowered* one degree Centigrade. From this definition it is plain that it takes 100 calories to raise 1 gram of water from 0° to 100° C., 500 calories to raise 5 grams over the same temperature range, and so on.

If, instead of using the flame to heat the water, we develop the heat necessary to raise its temperature thru one degree by friction—as for instance by rotating in the water a paddle driven by a falling weight—we can find out how many foot-pounds or ergs of work are necessary to develop one calorie of heat. This has been done with elaborate precautions to avoid losses of energy and it has been found that 42 million ergs of mechanical energy are required to develop one gram-calorie of heat. This

quantity is called the *Mechanical Equivalent of Heat.* Using English units, we may say that it takes 778 foot-pounds of work to raise one pound of water thru one degree Fahrenheit.

2. Specific Heat.—Very accurate experiments have been made to find out the number of calories required to raise one gram of other substances than water thru one degree Centigrade. It takes, as we have already indicated, $\frac{1}{11}$ (.09) calories to raise one gram of copper one degree; $\frac{1}{30}$ (.03) calories to raise one gram of mercury one degree, and so on. The number of gram calories required to raise one gram of any substance thru one degree Centigrade is known as the *Specific Heat* of that substance. We list here the specific heats of a few important substances.

Aluminum	.218	Iron	.113
Brass	.094	Lead	.0315
Copper	.095	Mercury	.033
Glass	.2	Silver	.0568
Ice	.504	Zinc	.0935
	Water 1.00		

Let us represent specific heat by *Sp. h.* It is clear from the definition of specific heat that the number of calories given out or absorbed by a mass m grams of a substance in falling or rising thru one degree Centigrade must equal $m \times Sp.\ h.$ calories and that the total quantity of heat given out or absorbed by the mass m grams in falling or rising from $t_1°$ *to* $t_2°$ must equal $m \times Sp.\ h. \times (t_2° - t_1°)$ or the mass times the specific heat times the change in temperature.

We can easily understand the method usually employed in finding specific heats. We know that if a hot

body be brought into contact with a colder one, heat will be conducted from the hot body to the cold one until both bodies have the same temperature thruout. The hot body loses a certain amount of heat energy and the cold body gains a certain amount. Now, by the general principle of the conservation of energy, these two amounts of heat, that is the heat lost by the hot body and that gained by the cold body, must be equal, if none of the heat goes into some third body. Speaking in terms of calories, we can say that whenever two bodies with different temperatures are brought together, the number of calories lost by the hot body equals the number of calories gained by the cold body after both have come to the same temperature provided there is no loss to a third body.

Suppose now we take a weighed piece of copper of mass M_c which has been held in free steam so long that we know its temperature to be 100° C. thruout and put it into a weighed quantity of water of mass M_w which has a thermometer dipping into it. Let the vessel containing the water be so arranged that no heat can escape from it. After a minute or two copper and water will have come to the same temperature which can be read on the thermometer. Then we know that the number of calories lost by the copper in falling from 100° C. to the temperature of the mixture must equal the number of calories gained by the water in rising from its original temperature to the temperature of the mixture. The number of calories gained by the water can readily be calculated by multiplying its mass into its specific heat into its *rise* in temperature, and the number of calories lost by the copper can be represented by the product of the mass of the copper into *its* specific heat into its *fall* in temperature. If we represent the original tempera-

ture of the copper by T_c, of the water by T_w, and the final temperature of the mixture by Tm, we can write the equation:

calories gained by water = calories lost by copper.

$$M_w \, Sph.h_w \, (T_m - T_w) = M_c \, Sp.h._c \, (T_c - T_m)$$

In this equation all the quantities are known from the experiment except the specific heat of the copper which we wish to find. We therefore write the equation $Sp.h._c = \frac{M_w \, Sp.h._w \, (T_m - T_w)}{M_c \, (T_c - T_m)}$ in which of course $Sp.h._w$ equals one. $(T_m - T_w)$ is the rise in temperature of the water and $(T_c - T_w)$ is the fall in temperature of the copper.

3. The Calorimeter.—The method just described for measuring heat quantities is known as the "Method of Mixtures." The work must be carried out, as we have said, in a vessel from which no heat can escape. A vessel constructed to meet these conditions is known as a *calorimeter*. Ordinary calorimeters are constructed of two cylindrical polished metal cans, one smaller than the other, made so that the smaller one, hanging freely from an asbestos or wooden cover, fits inside the larger, leaving an ample air space all around. The mixture is made in the small inner can which is provided with a lid with a hole in it thru which a thermometer and stirring rod can be passed. The layer of still air around the inner can prevents any large loss of heat by conduction or convection while the polish on the can very much reduces the loss from radiation since the radiant heat waves of which we spoke in a recent paragraph are, to a large extent, reflected from polished metal surfaces without being absorbed.

A better arrangement is one in which the space be-

tween the cans is a vacuum, since then there can be no loss at all by either conduction or convection except thru the cover. The ordinary thermos bottle is made in this way. It consists of a double walled glass container with the air exhausted from between the walls. The inner walls are silvered like mirrors to reduce the loss from radiation as much as possible. It is impossible to construct any calorimeter in which the heat losses by convection, conduction and radiation shall be exactly zero, but one can be made from which the rate of loss is so slow that no account need be taken of it in ordinary experiments extending over a few minutes.

Let us here consider a single problem. (1) What will be the temperature of the mixture resulting from mixing 100 g. of water at 90° with 200 g. of water at 10°?

If T_m represents the temperature of the mixture we have: heat in calories lost by hot water = heat in calories gained by cold water.

$$200\ (T_m - 10^\circ) = 100\ (90^\circ - T_m)$$

Multiplying out and solving for T_m we find $T_m = 36\frac{2}{3}^\circ$. In this problem no account is taken of the heat capacity of the calorimeter.

4. Changes in State.—We are now ready to study the changes in state produced by heat. If, on a cold day, we take some ice from outdoors, that ice will have the temperature of its surroundings. Suppose that temperature to be —10° C. If we heat the ice in a vessel containing a thermometer, we shall find that the temperature will rise to 0° C. at which temperature the ice will begin to melt. No matter how much we increase the rate of heating, we will now find it impossible, if we stir the mixture vigorously, to raise the temperature of the mix-

ture higher than 0° C., until all the ice is melted. As soon as the ice is all gone, the temperature of the water will begin to rise. During the time that the ice was melting, a large amount of heat apparently disappeared, since no temperature change resulted from its absorption. In other words there was no increase in the *kinetic* energies of the molecules of the mixture due to the absorption of this heat. Since, however, energy cannot be destroyed, it seems fair to suppose that the heat which disappeared is in the mixture in the form of *potential* molecular energy.

It is clear that the molecules of liquids are much more free to move about and to separate than are the molecules of solids. The molecules of a substance in the liquid state are therefore probably farther apart than they are in the same substance in the solid state. It must take energy to bring about this separation and, in our experiment, the necessary energy was just that which apparently disappeared in the heating. We therefore suppose that the energy is present in the liquid in the form of energy of separation of the molecules, that is, in the form of potential energy.

It necessarily takes a certain definite amount of heat energy to bring about this separation in a given amount of matter—that is to melt the given amount changing it from a solid to a liquid. If now we lower the temperature and allow the liquid to return to the solid state, we may expect that the energy previously absorbed and stored as potential energy will be liberated and reappear as heat energy, since when the liquid becomes a solid, its molecules fall closer together again. This liberation of heat on solidifying is in fact found to take place and accurate experiments have shown that the number of calories given out in solidifying is exactly equal to the

number of calories taken up by the same mass in melting.

5. Heat of Fusion.— The number of calories taken to melt a gram of any substance without raising its temperature is called the *heat of fusion* of the substance. It is very different for different substances, but always exactly the same for the same substance. We can easily find the value of the heat of fusion of ice by the method of mixtures. Suppose that when we put 150 grams of ice at 0° C. into 500 grams of water at 80°, the temperature when the last of the ice melted was 43.1°. The heat given out by the water has in that case been $500 \times (80 - 43.1) = 18{,}450$ calories.

Some of this heat was used to melt the ice and the rest of it was used to raise the temperature of the water resulting from the melting of the ice from 0° to 43.1°. If we represent the heat of fusion of ice—that is the number of calories required to melt one gram of ice without raising its temperature—by L then in this case the heat required to melt the ice has been $150 \times L$ calories. The heat required to raise the ice water from 0° to 43.1° has been $43.1 \times 150 = 6465$ calories. We have then $18{,}450 = 6465 + 150\ L$. Whence $L = \frac{18450 - 6465}{150} =$ 79.9 calories per gram. The most accurate measurements have shown that the heat of fusion of ice is very close to 80 calories per gram. This means of course that it takes 80 calories to melt 1 gram of ice without raising its temperature and that when one gram of water freezes, it will give out 80 calories of heat.

This last effect of heat liberation in the freezing of water is made use of by farmers who place large tubs of water in their vegetable cellars to protect the vegetables from freezing. The vegetables freeze slightly

below 0° and the water freezing first gives out enough heat in some cases to save the vegetables from freezing.

6. Melting Point.— The temperature at which the melting of a solid or the freezing of a liquid begins is found to be absolutely fixed and definite for all substances of a crystalline structure like ice. This temperature is called the *melting point* of the substance. With these crystalline substances the melting point temperature and the solidifying temperature are found to be the same and, as with ice, from the instant at which melting or solidification begins until it is entirely completed, the temperature of the substance remains constant. Glass, tar and waxlike or " amorphous " substances generally do not show any such definite melting points. When heated they get progressively softer and softer and gradually more and more liquid.

We have indicated that the general nature of the change that takes place when a body melts involves a separation of the molecules. We would therefore expect the body to expand when melting and to contract when solidifying. As a matter of fact most, but not all, substances do actually behave in this way. There are several important exceptions, however, among which is water. We can explain this by saying that the molecules of ice are arranged in such a way that they take up more room than when they are in the liquid state and consequently water expands on freezing. The amount of expansion of water on freezing is very considerable, the increase being about $\frac{1}{12}$ of the original volume.

This expansion has important practical consequences among which is the fact that as ice forms in water it floats on the top instead of sinking to the bottom. The

bursting of milk bottles and of water pipes in winter is caused by this expansion. The force exerted by freezing water is very great. Even heavy steel vessels can be burst asunder in this way. The breaking down of cliffs and rocks is caused largely by water freezing in the cracks.

Cast iron behaves like water in that it expands on solidifying. Hence its extensive use when it is desired to make a sharp casting in a mould. It is evident that the substance in the mould must expand on solidifying if the detail in the mould is to be brought out. Lead and most other metals contract on solidifying in accordance with the general rule. In the casting of type, lead alloyed with copper and antimony is used (type metal). The effect of the antimony in the lead is to produce an alloy that expands on solidifying. If sharp impressions are desired on ordinary metals, they must be obtained by stamping as in the case of gold, silver and copper money.

Since water in order to solidify must expand, we might expect that if we applied pressure to it so as to keep it from expanding, it would not freeze—and on the other hand, we might expect that if we applied pressure to liquid lead which must contract on freezing it would freeze more readily. These expectations are indeed realized in fact. Applying pressure to ice lowers its freezing point. This fact has some important practical applications. When a skater moves over ice, his skate blade exerts a considerable pressure on the ice under it. This pressure, except in the coldest weather, lowers the freezing point of the ice to a sufficient extent to melt the ice under the blade into a film of water over which the blade glides. This explains the ease of movement of a

skater over ice. The film of water refreezes instantly when the pressure is removed from it.

7. Vaporization.— We shall next consider the second change in state which heat can produce—the change from the liquid to the gaseous state. If we take a mass of water and heat it steadily, it will rise to 100° C. and then begin to boil, giving off bubbles of vapor from the bottom and interior of the water generally. If we have a thermometer in the water, we shall find that it is impossible to raise the temperature of the water above 100°. The application of additional heat does not raise the temperature—it only makes the water boil faster. As was the case with the melting ice, the heat energy which apparently disappears is, in fact, stored up in the steam as potential energy of molecular separation. If the steam is subsequently condensed, exactly the same amount of energy, the same number of calories, will be given out by each gram of steam in condensing, as was absorbed in the original boiling away of that gram of water. The number of calories of heat energy required to vaporize one gram of a liquid without any increase in temperature is called the *heat of vaporization* of the liquid. The *heat of condensation* of the vapor has exactly the same value as the heat of vaporization of the liquid, being defined as the number of calories of heat energy given out by one gram of the vapor when condensing without change in temperature. Different liquids have different heats of vaporization, but for any one liquid, the heat of vaporization is always exactly the same.

We can apply the method of mixtures to determine the value of the heat of vaporization of water. Suppose that when 17.6 grams of steam at 100° are con-

densed in 500 grams of water at 18°, the final temperature is found to be 39°. The amount of steam condensed (17.6 grams) can of course be found very easily by weighing the calorimeter cup before and after the steam has been condensed in it. The total amount of heat taken up by the water in the calorimeter has been $500 \times (39 - 18) = 10{,}500$ calories. This heat has come first from that given out when the steam condensed, and second from the hot water resulting from the condensed steam in falling from 100° to 39°. If we represent by Ls the heat of vaporization of water, that is the number of calories of heat given out by one gram of steam in condensing without change of temperature, then the heat given out by all the steam in condensing is clearly $17.6 \times Ls$ calories. Finally the heat given out by the water resulting from the condensed steam in falling from 100° to 39° is $17.6 \times (100 - 39) = 1073.6$ cal. Consequently we have $10{,}500 = 17.6\ Ls + 1073.6$ whence $Ls = 536$ cal. per gram. Accurate experiments have shown the heat of vaporization of water and the heat of condensation of steam actually to be very close to 536 calories per gram.

8. Boiling Point of Water.— The temperature at which what we ordinarily call the " boiling " of a liquid begins —that is the temperature at which bubbles of vapor begin to form in the interior of the liquid—is called the *boiling point* of the liquid. This point is definitely fixed for any given liquid except in so far as it, like the freezing point, is affected by changes in pressure. The effect of pressure on the boiling point is however much more marked than it is on the freezing point. We have defined the boiling point as the temperature at which bubbles begin to form in the interior of the liquid. The

bubbles contain saturated vapor of the liquid. It is clear that a bubble cannot exist in the interior of the liquid unless the pressure exerted by the saturated vapor in it is at least equal to the pressure of the air on the surface of the liquid. On the other hand, bubbles will begin to form thruout the liquid as soon as the pressure of its saturated vapor equals the pressure on the liquid surface. We can, then, give a more useful definition of the boiling point in this form: the boiling point is the temperature at which the pressure of the saturated vapor equals the external pressure.

We now see how easily and thru what wide limits the boiling point of a liquid can be varied. Any slight change in the pressure on the liquid surface affects the boiling point considerably. On top of Mt. Blanc in the Alps water boils at 84° C. In general a change in elevation upwards of 960 feet causes a drop of 1° C. in the boiling point of water. The boiling point in Denver, Colorado, is about 95° C. On the other hand under a pressure of 100 pounds to the square inch in a steam boiler water boils at 155° C.

It will be noticed that in evaporation, which takes place at all temperatures, the vapor forms only over the surface of the liquid, being made up of molecules whose velocities are relatively excessive; but in boiling, the vapor forms thruout the volume of the liquid.

We can apply a number of the facts which we have just been studying in the following simple problem:

Given 100 grams of ice at −10° C. Find how much heat is required to convert this ice into steam at 125° C., if the specific heat of ice is .5 calories per gram and the specific heat of steam is .46 calories per gram. To bring the ice to 0° C. will take $100 \times .5 \times 10 = 500$ calories.

To melt the ice without changing its temperature will take $100 \times 80 = 800$ calories. To raise the resulting water from 0° to 100° C. will take $100 \times 100 \times 1 = 10{,}000$ calories. To convert the water at 100° into steam at the same temperature will take $100 \times 536 = 53{,}600$ calories. To raise the resulting steam from 100° to 125° will take $100 \times .46 \times 25 = 1150$ calories. The total heat used is therefore $500 + 800 + 10{,}000 + 53{,}600 + 1150 = 66{,}050$ calories. It will be noted that by far the greater part of the energy of the steam enters it in the operation of vaporization.

9. Absorption and Liberation of Heat.—It is easily seen that when a solid is dissolved in a liquid and a solution is formed, a change of state has taken place in which a solid goes into a liquid form. We might therefore expect to find in the act of solution certain liberations or absorptions of heat similar to those observed during fusion. As a matter of fact heat effects are observed when a solid dissolves in a liquid. Sometimes, as is the case when common salt is dissolved in water, the temperature of the solution falls, showing an absorption of heat—in other cases, as when lye dissolves in water, the temperature rises, indicating a liberation of heat.

Solutions are found to have lower freezing points than the liquids used in making them. It is possible to get a solution of ordinary salt in water, the freezing point of which is —22° C. The action of the freezing mixture used in ice cream freezers is based on this fact. Suppose we have a salt solution of such strength that it freezes at —10° C., and we throw pieces of ice into it. The ice will melt in the solution. Each gram that melts will absorb 80 calories of heat from the solution and thus lower its temperature. If we continue to throw ice in, the process

of heat abstraction will be continued until the temperature of the mixture falls to its freezing point, in this case —10° C. If a container with pure water in it is dipped into this mixture, the water will be frozen.

We are here making use of the heat of fusion to get a lower temperature. A much more rapid effect can be produced by making use of the absorption of heat which accompanies vaporization. If liquid ammonia is allowed to expand freely out of a valve, it will vaporize into gaseous ammonia and in so doing will absorb very rapidly a large amount of heat. If the expansion takes place in a pipe surrounded with salt solution, the heat will be taken out of the salt solution, the temperature of which will be lowered. This is exactly what is done in artificial ice plants where gaseous ammonia is liquefied by applying to it a pressure of about 150 pounds to the square inch and then allowed to expand into the gaseous form at a pressure of about 30 pounds per square inch in a system of pipes contained in tanks of brine, the temperature of which is thereby reduced to about —10° C. Pure water held in oblong vessels in the brine is thus frozen. The expanded gas is carried back to the pressure pump and reliquefied. The heat which is liberated on liquefaction is carried away by cooling the pipes containing the liquid ammonia before allowing it to expand into the pipes in the brine tanks. Cold storage plants work on exactly the same principles, but the cooled brine is pumped thru systems of pipes into the various rooms of the storage house.

Before leaving the subject of heat we must briefly consider a few of the most important industrial uses made of heat energy in steam and gas engines. All the usable energy on the earth had its original source in

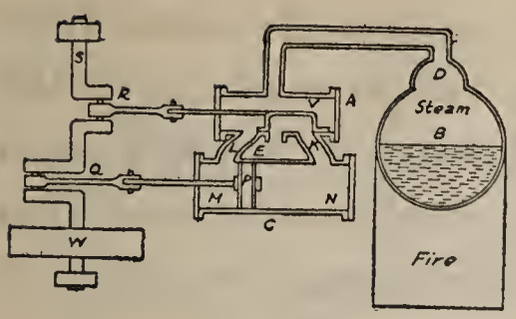

Fig. 47.

the heat energy of the sun. Coal, for instance, contains energy extracted from the sun's rays in the forests of the Carboniferous Age. Water power depends on the fall of water originally raised by the sun thru evaporation from the surface of the ocean, followed by elevation thru convection currents and precipitation as rain. The stored-up heat energy in fuels as in coal and gasoline is rendered available for our use in certain engines, the essential construction of which we shall now consider.

10. Reciprocating Steam Engine.—First let us examine the form of reciprocating steam engine invented by James Watt, of Glasgow, about the year 1768. Water is boiled in the boiler *B*, Fig. 47. The steam is collected under its own pressure in the steam dome *D*. A pipe leads from this dome to the valve chest *A* over the cylinder *C*. This chest has two pipes *K* and *L*, one leading to either end of the cylinder as shown, and a third pipe *E* communicating with the outside air. A sliding valve *V*, with the general shape shown, moves back and forth in the chest in such a way that at one end of its travel *K* is connected with *E*, while *L* is open into the chest and thru the chest into the steam dome, and at the other end *L* and *E* are connected and *K* put into connection with the steam dome. With conditions as shown in Fig. 47, steam from the dome will enter the space *M* and push the piston across to the other end of the cylinder. This, by means of the crank *Q*, will turn the shaft *S* one-half revolution, and in so doing will turn *R* one-half revolution and pull the valve *V* to the other end of the steam chest, thus connecting *L* and *E*, so that the steam in the

end of the cylinder may blow out, and at the same time connecting *K* and *V* so that live steam may enter the other end, *N*, of the cylinder. This live steam will push the piston back towards *M* and the continuous repetition of this operation will result in a continuous rotation of the wheel *W*.

In practice the steam is allowed to escape directly into the air only in cases where simplicity is a primary object, as in most steam locomotives. In stationary steam plants, the steam is nearly always discharged into a condenser in which the pressure is not more than one pound to the square inch. This of course adds 14 pounds to the effective pressure of the steam on the piston and thus improves the efficiency. Such engines are called *condensing engines* while those of the simpler type, which exhaust into the air are called *non-condensing engines*.

In very large, steadily running engines such as are used on steam ships, the steam usually is conducted from the first cylinder into a second cylinder and thence into a third and even into a fourth cylinder before being taken into the condenser. Each of these successive cylinders is larger in cross area than the preceding one and in each the steam expands farther than in the preceding one and thus gives up more energy and develops a higher efficiency. Such engines are called *compound engines* and are referred to as *double, triple* or *quadruple expansion* engines.

11. Steam Turbine.—All the engines described above are " reciprocating " in that they contain as the essential driving feature a piston which reciprocates, that is, moves back and forth. Several types of rotatory steam engines are in use which offer many advantages over the reciprocating type. These rotatory engines are called

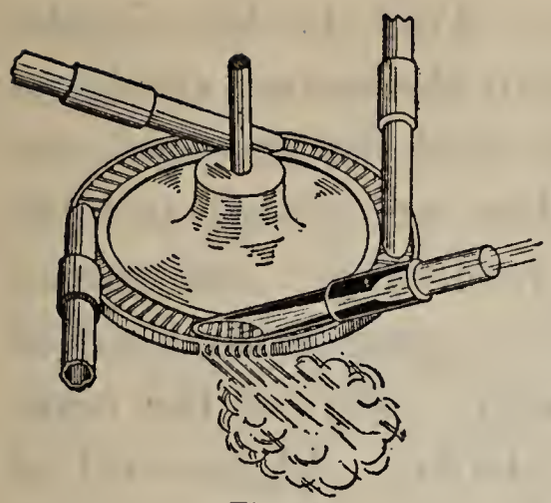

Fig. 48.

steam turbines. The live steam is directed thru nozzles onto curved vanes of metal set in the rim of a wheel. The steam drives the wheel around, more or less as wind drives a windmill. In practice, the steam after striking a row of vanes on one wheel is redirected by a set of stationary guide vanes to strike the vanes of a second wheel mounted on the same shaft, thus making use of a larger amount of the energy of the steam. Frequently the steam strikes on 14 to 16 rows of movable vanes before passing into the condenser. This type of engine occupies about $\frac{1}{10}$ of the floor space taken by a reciprocating engine of the same power. Many of the largest ocean liners and battleships have engines of this type and most of the modern city electric light plants also use turbines. Many of these turbines develop as high as 25,000 H. P. (See Fig. 48.)

12. Gas Engine.— The steam engine is an expansion engine but the gas engine such as is used in automobiles is an explosion engine. Practically all automobile engines are piston engines and operate on what is known as a four stroke cycle—that is a cycle in which there are four strokes of the piston to each power impulse. The cylinder has only one closed end. In or near this end, the head, there are two valves which are driven by cams on a driving shaft so that they open and close at the proper times.

On the first stroke of the cycle (Fig. 49) the piston moves to the left with the intake valve open. An explosive mixture of air and gas is thus drawn into the

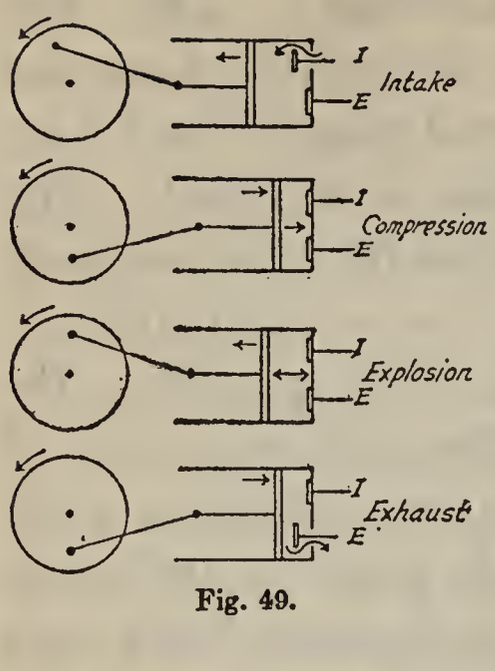

Fig. 49.

cylinder. This is the intake stroke. On the second stroke of the cycle both intake and exhaust valves are closed and the piston, returning, compresses the mixture against the head of the cylinder. This is the compression stroke. At the end of this stroke an electric spark passes in the compressed gas, exploding it. This explosion drives the piston back on the third stroke of the cycle. This is the explosion or power-stroke. Both valves still remain closed. On the fourth stroke the piston returns with the exhaust valve open and pushes out the burned gases. This is the exhaust stroke. The next following stroke is again an intake stroke and the cycle is then repeated. If there is only one cylinder, the engine must have a heavy fly wheel to carry it thru the three strokes where power is used up, to the one-stroke where power is developed.

The efficiency of a heat engine is defined as the ratio between the heat energy transformed into useful work and the total heat energy absorbed. On account of the fact that some heat must be thrown out at the exhaust, no heat engine can operate at very high efficiency. An ideal engine with no avoidable losses, if using steam at 190° C. and delivering water in the condenser at 40° C., would have an efficiency of 32%. In the very best types of triple expansion engines the actual efficiency attained is only 17% or thereabouts. In ordinary locomotives the efficiencies are about 6% to 8%. Gas engines, on account of working at much higher temperatures than steam engines, have higher efficiencies. Large gas

engines sometimes reach an efficiency of 37%, while the smaller ones used in motor cars frequently have efficiencies around 25% to 28%.

REVIEW.

1. What is a calorie?
2. What effect has pressure upon (a) the freezing point of ice, (b) the boiling point of water?
3. What is meant by the statement that all usable energy on the earth has its source in the heat energy of the sun?
4. What is a condensing steam engine? A compound engine?
5. Distinguish between an expansion engine and an explosion engine.

CHAPTER VIII

SOUND—ITS NATURE AND PROPAGATION

1. Common Source.—We are already familiar by experience with many facts concerning sound, hence the study of the science of sound which we are about to take up is certain to prove both interesting and useful. We shall consider first the exact nature of sound together with the modes by which sounds are produced.

Whatever sound we may hear, if we trace it to its source, we shall always find that it comes from some object in a state of rapid vibration. This vibration can usually be detected either by looking at the sounding body or by feeling it. For instance, when a bell is ringing, the rapid trembling movement of its edges can readily be felt against the finger. So with a tuning fork the prongs of which can be seen to tremble or vibrate as long as it is producing a sound. Again the trembling of a string in a piano can easily be observed when the string is sounding. If this trembling is stopped, either in the case of the string, the fork or the bell, the sound instantly ceases.

A body can be caused to give out sound either by being struck or else by being rubbed over some other body—the striking or rubbing serving to set up the necessary trembling motion. The bell for instance is struck sharply by the clapper, the tuning fork is rapped against a wooden or rubber block, and the piano string is hit with a felt-covered hammer driven by the piano key. As an example of rubbing, consider the string of

the violin which is made to sound by rubbing it with a rosined bow; or take the case of the brakes on a street car, which squeak sharply when set.

If the sound is to continue for any length of time after the blow on the body ceases, it is evident that the body must be an elastic one, that is, as we have previously stated, one which after being distorted tends to return to its original shape. On its first return, it will, on account of its inertia, overshoot its true position a bit and will continue to vibrate back and forth for some little time before it stops in its original place. During this vibration the sound continues to be made. This is the case with the tuning fork and other objects that we have just spoken of. If the body that is struck is not elastic, one short sound only—the sound of the blow—will be produced. This is the case if a block of lead is struck with a hammer, or if a piece of soft putty is thrown on the ground. Again if the body, as for instance the tuning fork, instead of being sharply struck is pushed to one side and made to come back slowly to its original position, no sound at all is produced. It seems, then, that for the production of sounds the body must move rapidly.

We can, consequently, in summing up, say that sustained sounds are produced by rapidly vibrating elastic bodies. Such bodies can be stimulated to produce sounds by being struck or rubbed. A single sound—not sustained—like an explosion—can be produced by any body making a single very rapid movement. A bullet flying thru the air produces for example a single very sharp, cracking sound.

2. Transmission of Sound.—Having now established that sounds arise from rapidly moving bodies, the next

point is to find out how the disturbances produced by the motion of the body get from the source to the ear which perceives the sound. This point can easily be settled by experiment. If we put an electric bell which is ringing continuously on a thick pad of cotton wool under the receiver of an air pump and slowly exhaust the air, we find that, as the air becomes less and less dense, the sound of the bell becomes fainter and fainter—until, when a good vacuum is reached, absolutely no sound at all can be heard, altho the hammer can be seen to be moving as vigorously as ever. If now the air is readmitted, the sound begins to come again reaching its original intensity when all the air has gone back into the receiver. It is clearly evident from this that the disturbance produced by the moving gong has been carried to the ear thru the air.

Our studies of air as a gas have shown us that when air is compressed, it tends to expand again to its original volume—that is to say that air is distinctly elastic. This being the case, we might expect the vibrating bell to produce a similar vibration in the elastic air around it, which vibration, entering the ear might produce that effect on the nerves of the ear which we call a sensation of sound. This is exactly what does happen. It is clear that the same effects will be produced when any other medium than air surrounds the bell provided only that the new medium be sufficiently elastic. If, for example, when swimming, the head is held below the water, sounds can be heard with great distinctness coming thru the water. Submarine boats have frequently been detected by listening for the sounds of their engines under the

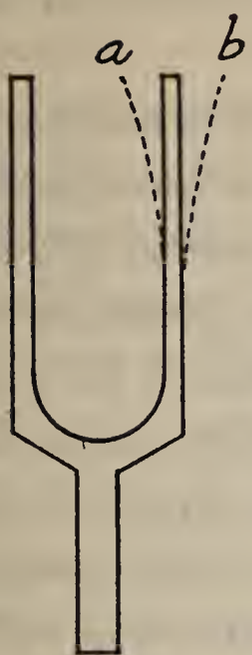

Fig. 50.

water. Similarly the sounds of train wheels are transmitted over great distances thru the steel rails of the track. We can say in general, then, that the disturbance produced by a very rapidly moving body can be transmitted thru, or to use more scientific language, can be propagated in, any elastic medium which may surround or be in contact with the moving body.

3. The Tuning Fork.—Now let us consider somewhat in detail how the sound disturbance is transmitted thru the air. After being disturbed, a tuning fork (Fig. 50), which is made of highly elastic steel, vibrates back and forth with great regularity of motion. Suppose that the fork is surrounded by air and that the end of the prong (we will pay attention to only one of the prongs) is at *a*. Let the prong move across from *a* to *b* in a very short length of time, say $\frac{1}{200}$ of a second. In this short time the air in front of the prong has not time to flow around the prong to the other side. It is caught by the prong and sharply compressed. As soon as the prong starts back from *b* toward *a* this compressed air will expand and in expanding will compress the air all around it, thus producing a spherical shell-like compression. This compression is produced because the air inside expands so quickly that the air outside does not have time to flow away. The spherical shell of compression will in turn expand and compress the air just beyond itself and so on, the result being that a spherical shell of compression spreads out from the end of the prong as a center at a high rate of speed. When this compression enters the ear it

affects the ear drum and thus produces a sensation of sound.

Returning to the tuning fork, let us note what happens when the prong goes from *b* to *a*. This movement takes place so quickly that something of a hole is left in the air behind the prong—that is to say, the pressure in the space behind the prong as it moves from *b* to *a* is much reduced. Such a space with a reduced pressure in it is called a "rarefaction"—as being the opposite of a "compression" or space containing an excess pressure. The air immediately around this rarefaction will expand into the "hole" forming a spherical shell of reduced pressure, that is a spherical rarefaction, around it. The air just outside this spherical rarefaction will in turn expand into it and so on, the net effect being the spreading out of a spherical rarefaction from the end of the prong as a center at the same rate that the condensation spread out. It will be noticed that this rarefaction follows immediately behind the compression and that a second compression caused by the prong moving from *a* to *b* a second time will immediately follow the rarefaction, and so as long as the fork continues to vibrate, a compression will travel out each time the prong comes from *a* to *b*, and between every two compressions will be a rarefaction produced by the prong going from *b* to *a*. If the prong moves across 200 times a second, then in each second 200 spherical compressions will follow each other out into the air.

These rapidly alternating compressions and rarefactions striking on the ear drum produce a definite continued sensation of a sound which a musician would recognize as being close to the G sharp, next below the middle C of a piano. Such a succession of con-

densations and rarefactions is called a *train of sound waves.*

One condensation taken together with the one rarefaction immediately following it is said to make up a "single wave" of the sound, and the distance from one condensation to the one next following is called the "wave length" of the sound. Waves of this kind made up of compressions and rarefactions are called *compressional waves,* or, since the motion of the particles of the medium brought about by the wave is back and forth along the line of propagation, *longitudinal waves.*

4. Speed of Sound Waves.—The rate at which waves of this sort spread out in air has been measured by noting with a stop watch the interval between the arrival of the flash and of the sound of a cannon fired at a known distance. When there are no disturbances due to wind or other motions of the air and the air is dry and at 0° C. temperature, the sound has been found to travel 331.3 meters in a second. This is equal to 1087 feet per second (about 12 miles in a minute) at 0° C., or to 1126 feet per second at 20° C., which is an average outdoor temperature for warm weather. The speed has been measured also in water where it was found to be 1400 meters in a second and in iron, where it was found to be 5100 meters per second. In general, sound travels about 4 times as fast in liquids as in air and about 12 times as fast in highly elastic solids as in air.

It will be noticed that the rate of transmission of the wave depends on the temperature. The effect of temperature in the case of air is a very important one. We have said the speed in air at 0° C. to be 331.3 meters per second. For each rise of 1° C. the velocity increases 60 centimeters, that is .6 meter per second. From this we

can calculate that for each rise of 1° F., the velocity must increase 1.1 feet per second. This change is so large that it must always be taken into account in calculations involving the speed of sound.

Returning to our tuning fork, we now know that one second after the prong makes its first movement from *a* to *b*, the first compression will have traveled (at 20° C.) 1126 feet away from the fork. In this space between the fork and the 1126 foot mark, there will be 200 compressions alternating with 200 rarefactions, that is to say, there will be 200 waves in the space. If we represent the number of waves given out in a second, which is known as the *frequency* of the sound, by *n*, the distance traveled by the wave in one second, that is the *velocity* of the wave, by *v* and the length of a single wave (from one condensation to the next condensation) by *l* it is clear that $v = nl$ or $l = \frac{v}{n}$. This must be so because the total wave train produced in a second covers *v* feet and contains in it *n* waves. The length of one of these waves must consequently be $\frac{v}{n}$.

5. Reflection of Sound.— Let us now suppose that this train of waves from the fork strikes against a solid wall. When the air against the wall is compressed, it cannot expand forward being checked by the wall. In that case the necessary expansion will take place backward, and the first compression, followed in turn by the rarefaction and by the following waves, will travel back away from the wall at the same rate that it came up. This is known as *reflection* of sound. Reflection always takes place when a wave strikes on the boundary of a medium of different density from that in which it has been traveling. When we are speaking in a closed room, our

auditors hear not only the original sound produced in our throats but also the sounds reflected from the walls. It is because of this that it is so much easier to hear and to make ourselves heard in a closed room than in the open air. The reflections, of course, follow the original sound and thereby prolong it. In a poorly designed, large hall the reflections of one word may become confused with the sound of the next word spoken. In such halls, which are said to have " poor acoustic properties," it is very difficult to understand what a speaker says. These effects are always more marked when the hall is empty than when the seats are filled with people, for the people form a large number of irregular reflecting and absorbing surfaces which absorb and disperse the energy of the reflected waves.

The ear is so constructed that it cannot distinguish or separate clearly two sounds which follow each other at an interval of less than $\frac{1}{10}$ of a second. If therefore a reflected sound comes back to the ear within $\frac{1}{10}$ of a second of the time the original sound arrived at the ear, the two sounds will be heard as one; but if the reflection returns $\frac{1}{10}$ second or longer after the original sound, it will be heard as a separate and distinct sound. This explains the production of the *echoes* frequently heard among the hills or in very large buildings. To get an echo we must have a large reflecting surface 55 feet or farther from the listener, if the sound be produced near the listener. This distance is necessary because if the total space covered by the sound is less than 110 feet, the reflection will get back within $\frac{1}{10}$ second and will not be clearly perceived.

If there are a number of reflecting surfaces, or if the

sound is reflected back and forth several times, as between two hills, then a number of successive echoes called *multiple echoes* may be heard. If the reflecting surface happens to be regular and more or less spherical, such as the interior of a dome, the reflected compressions may be bent into such a form that each one converges on some point near the center of the dome. A person stationed at this point will hear clearly sounds which are much too faint for persons nearer the source to perceive. This is due to the concentration of the energy by the converging reflection. So-called "whispering galleries" work on this principle.

6. Law of Inverse Squares.—We have noted that the sound wave is propagated outward from the source at the rate roughly of 1100 feet per second in the form of concentric spherical shells. It is evident that the total energy contained in the original first compression made by the fork is at the end of a second distributed over the surface of a sphere with a radius of about 1100 feet. The total amount of energy which would enter the ear of a listener, let us say thru a tube of 1 sq. cm. area, at this distance is seen to be exceedingly small. As the listener came closer and closer to the source, the amount of energy which would enter his ear thru the square centimeter of area would become larger in proportion as the areas of the spheres around the source on which his ear lay became smaller. For example, when the area of this sphere was reduced to one-half, the amount of energy on a unit area of the sphere would clearly be doubled, since the total energy on all the spheres is the same. The sound would therefore become louder or more intense as the listener approached the source and this increase in intensity would not be directly with the

decrease in distance, but directly with the decrease in area of the spheres of which the distances are radii.

If the first distance from the source is r_1 and the second distance from the source is r_2, the area of the sound compression at r_1 will be $4\pi r_1^2$, its area at r_2^2 will be $4\pi r_2^2$. Now the intensity, say I_1 at r_1, will be to the intensity I_2 at r_2 in the inverse proportion of these areas, that is, $I_1 : I_2 : : 4\pi r_2^2 : 4\pi r_1^2$, or $\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$—that is to say, the intensities are inversely proportional to the squares of the distances from the source. We are here taking the intensity of the sound at any distance to be measured by the total amount of energy in each square centimeter of its wave front at that distance. This law connecting the intensity of a disturbance with the distance from the source is a general law of very wide application. It applies to every case where a disturbance passes out thru an elastic medium in the form of waves. It is called the Law of Inverse Squares. If the distance from the source is doubled, the intensity is reduced to $\frac{1}{4}$; if the distance is trebled the intensity is reduced to $\frac{1}{9}$ and so on. This law has its direct applications in light and in magnetism and electricity as well as in sound. It will be noted that the law applies accurately only when the wave can spread out unimpeded in all directions so as to have a spherical form. If the wave is confined by a tube or other device, it may travel for relatively great distances with small loss in intensity. This effect is illustrated in speaking tubes and megaphones.

The intensity of the sound at the source depends of course on the total amount of energy put into each compression. This will also affect the intensity at any given distance. As the energy of the compression equals the

work done on it in making it, this energy is clearly proportional, in the case of the tuning fork, to the distance the prong of the fork moves back and forth, that is to the amplitude of motion of the fork. We can say, therefore, that the intensity of a sound as perceived by a listener depends on two things, (1) on his distance from the source of the sound, and (2) on the amplitude of vibration of the source.

7. Pitch.—We have said that a musician would recognize the sound of this fork about which we have been talking so much as close to the note G sharp in the musical scale. If we took a fork of such size and material that it vibrated 256 times a second instead of 200 times, the musician would say that the sound produced was middle C in the scale. This difference in the apparent " height " of sounds, depending on the number of waves sent out in a second, is referred to as difference in *pitch.* We can of course all distinguish differences in pitch and we all know what is meant by the term " pitch." The thing to be noted here is that differences in the pitches of sounds are due entirely to differences in the rates of vibration of the sources of the sounds—that high-pitched sounds come from bodies which are vibrating very rapidly, while low-pitched sounds come from bodies vibrating less rapidly.

It has been found by experiment that the speed of sounds is entirely independent of their pitches. It follows that after two forks of frequencies of 100 and 200 respectively have been sounding for one second, the first compression from both forks will be 1126 feet away from its starting point and in this space there will be 100 waves from the one fork—each wave having a length of 11.26 feet—and 200 waves from the other fork—each of

these waves having a length of 5.55+ feet. It is plain from this that the waves of high-pitched sounds are shorter than those of low-pitched sounds.

The pitch that we judge a sound to have when we hear it, depends on the number of waves per second which actually enter the ear. If 256 waves enter the ear in a second, we say that is "middle C"; if 512 waves enter in a second we say that is an octave above middle C.

The number of waves entering the ear is necessarily the same as that given off by the fork in the same time only when both fork and observer are stationary. For suppose the fork while sounding with a frequency of 200, moves forward 100 feet in one second. Suppose the temperature to be such that the velocity of sound is 1100 feet in a second. Then at the end of the second of time the first compression will be 1100 feet away from the original position of the fork. But the fork will have moved up 100 feet so that the 200th compression is only 1000 feet behind the first one. Into this 1000 feet we have crowded 200 waves. The length of each of these waves is clearly only 5 feet instead of the length of 5.5 feet which each wave would have had if the fork had not moved. An ear at the 1100 foot mark will therefore be struck by 5 foot waves and will get exactly the impression of pitch that would be received from a stationary fork with a frequency of 220 instead of the true frequency of 200.

Exactly similar reasoning will show that if the fork moves away from the observer, the apparent pitch will drop. The statement that the apparent pitch of a sound depends in this way on the relative motion of the source and the observer is known as *Doppler's Principle.* A sharp rise in pitch due to this effect can be perceived by

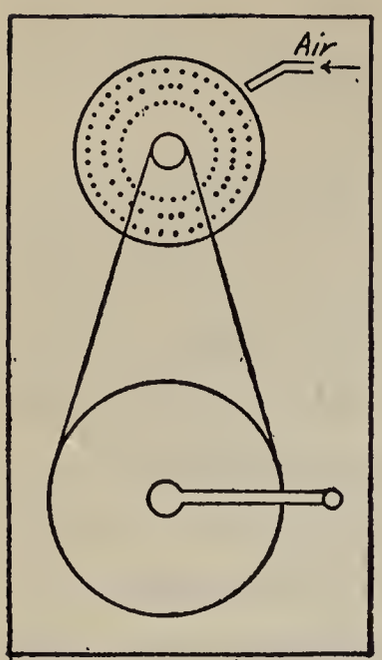

Fig. 51.

a passenger in a train when passed by a locomotive sounding its whistle while running in the opposite direction on the other track. The relative speed is in this case possibly very high—perhaps 120 miles per hour or more—and the effect is distinct.

This principle applies to any form of wave motion—to light, for example, as well as to sound. It is interesting to know that it is by a direct application of this principle, when referring to light, that the speeds of the various stars are determined.

8. Musical Sounds and "Noise".—We have now shown that the pitch of a sound depends on the number of vibrations made by the sounding body in a second and therefore on the length of the wave produced in the air by the sounding body. A great many sounds have, however, no definite pitch. When a load of bricks is dumped on a hard pavement, we find it impossible to assign any pitch to the resulting sound. We say that such a sound is merely a " noise." We can get a clearer idea of this matter by the use of an instrument known as a " siren." This is a metal disk 12 or 15 inches in diameter which has three concentric rows of holes punched in the edge as shown in Fig. 51. The outer row has 40 holes equally spaced around the edge—the next row has 30 holes very irregularly spaced—the third row has 30 holes regularly spaced. This disk can be rotated at a high rate of speed. A glass nozzle is supplied thru which a stream of air can be directed against any one of the rows of holes. When the jet is directed against the first row—the one with

40 regularly spaced holes—40 puffs of air come thru for each revolution of the disk. When the disk is speeded up, these puffs blend into a pleasant sound with a definite musical pitch. If now the nozzle is changed to the third row so that we get 30 regularly spaced puffs of air for each revolution—the sound has still a definite musical pitch but this pitch is much lower than before, just as we would expect. If finally we set the nozzle on the second row, where the thirty holes are arranged irregularly, the total number of puffs for each revolution is the same, yet the sound given is unpleasant and produces no impression whatever of definite pitch. It is an irregular noise.

This experiment furnishes us with the means of distinguishing musical sounds from noises. If a sound has a definite pitch, that is, if its wave train has a definite wave length, then it is called a *musical sound.* The effect of such sounds on the ear is generally pleasant unless the pitch is very high. If the sound has no definite pitch, if its wave train is made up of waves all of different lengths, it is called a *noise.* Noises are generally annoying in effect and irritating.

The human ear has a limited range of perception. If the pitch of a sound lies above a certain limit, or below a certain limit, the sound becomes inaudible. The range of perception is not the same for all ears and varies with age, being narrower for older people. Most persons can distinguish sounds arising from frequencies between 30 vibrations per second and 40,000 vibrations per second, altho it is difficult to judge accurately of pitches with frequencies over 4000 per second. Some people can hear sounds due to frequencies of only 16, while others can hear sounds as high as 50,000. The ordinary $7\frac{1}{3}$

octave piano has a range from its lowest note to its highest of 27.5 vibrations per second to 4220 vibrations per second.

REVIEW.

1. Describe the compressions and rarefactions produced by a vibrating tuning fork.
2. How are echoes produced?
3. State and explain Doppler's Principle.
4. Distinguish between a musical sound and a "noise."

CHAPTER IX

MUSICAL SCALES AND INSTRUMENTS

1. The Musical Scale.—We shall now study the relation in pitch existing among the notes of the so-called musical scale. This scale is a sequence of notes picked out on the general principle that they harmonize or sound well together. We can begin our study by the use of a new metal disk on the siren of Fig. 51. This disk has four rows of regularly spaced holes. The outer row has 48 holes, the next row 36 holes, the next 30 and the last 24. If when this disk is rotating steadily, we puff air thru the four rows of holes in succession, we shall find that the easily recognizable sequence *do, mi, sol, do*^{1} is produced. This sequence can of course be immediately recognized by a person with any musical training. If now the disk is made to go faster, the pitches of all the notes will rise but the character of the sequence will remain the same. It will still be *do, mi, sol, do*^{1}. This experiment brings out two important facts (1) that the sequence *do, mi, sol, do*^{1} is produced by notes whose frequencies are in the ratio of 4, 5, 6, and 8; and (2) that the sequence depends not on the absolute pitches of the notes but on the existence of these particular ratios among the pitches.

The complete natural musical scale is built up of eight notes, the vibration numbers of which are found to bear certain definite ratios to one another. This scale can have its beginning note on any pitch. It is only necessary that the succeeding notes of the scale shall have

vibration numbers bearing the proper ratios to this beginning note and therefore to each other. The ratios of the frequencies of the successive notes in the major diatonic scale, which is the one most commonly used in musical work, are, referring each note to the beginning note:

No. 1 C.	No. 2 D.	No. 3 E.	No. 4 F.	No. 5 G.	No. 6 A.	No. 7 B.	No. 8. C.
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

The letters usually applied to these notes to specify them are shown at the top of the diagram. The significance of this series of ratios can best be brought out by an example. Suppose we select a pitch of 256 vibrations per second for C. What will be the proper pitches for the other notes of the major diatonic scale built up on this C?

$$D = \frac{9}{8} \times 256 = 288 \qquad G = \frac{3}{2} \times 256 = 384$$

$$E = \frac{5}{4} \times 256 = 320 \qquad A = \frac{5}{3} \times 256 = 426\frac{2}{3}$$

$$F = \frac{4}{3} \times 256 = 341\frac{1}{3} \qquad B = \frac{15}{8} \times 256 = 480$$

$$C' = 2 \times 256 = 512.$$

2. Intervals and Chords.—It will be understood that the notes comprising the scale were selected originally because they, of all notes tried, sounded best together in various combinations. Years after the scale was built up, investigation showed that the ratios existing between the successive notes, all referred to the first note, were those listed above. Further tests show that the two notes which sound best of all together to persons with good musical ears are those whose frequencies have the simplest ratio, namely, the octave 1 to 2. The next smoothest sounding combination is that of C and G where the ratio is again a very simple one, 2 to 3. This

interval, being between the first and fifth notes of the scale, is called a *fifth.* Other smooth sounding pairs are G and C^1, the ratio of which is 3 to 4, called a *fourth,* and C and E with the ratio 4 to 5, called a *third.* It will be noticed that the smooth combinations are invariably those with simple ratios between their frequencies.

Turning now to combinations of three notes, it is found that C E G sounded together give a remarkably pleasing effect. These notes have frequencies in the ratios 4.5.6. Any three notes having these ratios (4.5.6.) among their frequencies are said to compose a *major chord.* The major diatonic scale is built up of three major chords, (1) C E G, called the *tonic chord* furnishing as it does the beginning note of the scale, (2) G B D^1 called the *dominant chord* passing over into the next octave, and (3) F A C, called the *subdominant chord.* It will be seen that these three chords taken together give all the notes of the major scale.

We have said that by the use of the ratios $1, \frac{9}{8}, \frac{5}{4}$, etc., a major scale can be built up on any pitch as a base. We have already calculated the pitches for the successive notes built up on C = 256 as a base. If now we take D = 288 from this C scale and build up a major scale on that (D = 288) as a base, we get the following values:

Ratios		1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Scale of D = 288		D 288	E 312	F 350	G 384	A 432	B 480	C^1 540	D^1 576
Scale of C = 256	C 256	D 288	E 320	F $341\frac{1}{3}$	G 384	A $426\frac{2}{3}$	B 480	C^1 512	D^1
Ratios	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2	

It will be seen that altho some of the frequency values for the higher notes coincide in the two scales, most of them do not coincide. This means that if we wish to construct an instrument like a piano which will play accurately in both scales, we must put in 11 keys instead of 7 to cover from D to C^1. If the piano were made to play accurately in all the scales which can be built up on the successive notes, some 80 keys would be required in each octave. The difficulty of playing on such an instrument would be enormous. In all instruments of the piano type therefore a compromise scale is used, known as the *evenly tempered scale.* In this scale the interval of the octave (1 to 2) is divided into 12 equal intervals, represented by the intervals between the eight white and five black keys of the piano. This scale is not an accurate major diatonic scale, since the ratios are not those listed above but the differences from the true scale are so small that very few persons can detect them. The comparison with the scale of C is shown below:

	C	D	E	F	G	A	B	C^1	
True	256	288	320	$341\frac{1}{3}$	384	$426\frac{2}{3}$	480	512	Vibrations per sec.
Tempered	256	287.4	322.7	341.7	383.8	430.7	483.5	512	" " "

The general relation existing between true and tempered scales can be more clearly brought out by expressing the frequency ratios of the successive notes referred to the first note in both scales in decimal fractions as below:

	C	D	E	F	G	A	B	C^1
True Scale	1.000	1.125	1.250	1.333	1.500	1.667	1.875	2.000
Tempered Scale	1.000	1.122	1.260	1.335	1.498	1.682	1.888	2.000

The agreement is seen to be very close. It will be remembered that the full tempered scale has 13 notes in

an octave with 12 equal intervals called semi-tones between them. As the intervals are all equal, a tempered scale can be run up from any key by taking an additional semi-tone for each successive note. Pianos can therefore be readily played in any key. It will be noted that an instrument like a violin having no keys or fixed frets can be played in a true scale or in a tempered scale, at choice, by a person having a sufficiently accurate ear.

If a single instrument is playing alone it makes very little difference in the general effect, except in "brilliance," whatever may be the absolute pitch of the note on which the scale used begins. When a number of instruments are playing together in an orchestra, it is however of foremost importance that the absolute pitches of the notes in the scales used should be identical for all the instruments. Otherwise very unpleasant discords will be produced. Certain standard pitches, usually for the first C next above the middle C on a piano, called C″, have therefore been adopted by various musical associations. The C″ used in scientific work is the one that we have been referring to so far. This scientific C″ = 512 vibrations per second. The French Standard "Diapason Normal" of 1859 C″ = 522, is however coming into almost universal use among organ and piano manufacturers in Europe and America. Other standard pitches in use are Concert Pitch (C″ = 546), Society of Arts (C″ = 528), Tonic Sol-fa (C″ = 507) and the Philharmonic (C″ = 540). The tendency has been to raise the standard pitch with the passage of years in order to make the music more "brilliant." In the times of Handel (1750) A had a frequency of 422.5 where at the present day 460 is often used. An old A

fork, of date about 1650, found in Paris, has a frequency of only 374 vibrations per second.

3. Resonance.—We may now turn to the subject of musical instruments and similar devices for producing sustained sounds. We shall first consider the *organ pipe* but before entering directly on the study of it, certain preliminary experiments must be described. We have already explained how a sound wave is reflected when it strikes a medium of density different from that in which it was previously traveling. We might easily imagine that when the source is sounding continuously, these reflected waves on their return could, under certain conditions, interfere with the waves still coming off from the source. The result of this interference might be either re-enforcement where two compressions came in the same place, or a reduction when a reflected rarefaction coincided with a condensation freshly given off by the fork. The actual occurrence of this effect of interference can be shown by the following experiments.

Take a tall glass cylinder. Hold over its mouth a vibrating tuning fork. The fork being held in the hand will produce only a weak sound. Slowly pour water into the cylinder thus reducing the length of the air column in the tube. Finally, as the length becomes less, a point will be reached at which the sound of the fork is very strongly re-enforced so that it can be heard all over a large room. This re-enforcing effect is called *resonance.* It is caused, as we indicated in the last paragraph, by the combination of the waves reflected from the water in the tube with those just being given off by the fork. The length of air column necessary to produce this effect will be found to be shorter, the higher the pitch of the fork used. If a fork making 256 vibrations per

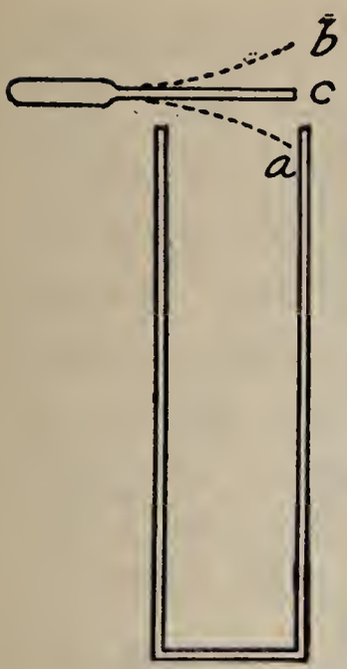

Fig. 52.

second is used with the air at such a temperature that the velocity of sound is 1126 feet per second, then measurement will show the necessary length of air column to be about 1.1 feet. The wave length of this sound is $\frac{1126}{256}$ or about 4.4 feet. The length of air column necessary to produce resonance in this case is seen to be $\frac{1}{4}$ of the length of one wave. Further experiments would show that no matter what may be the pitch of the fork, the length of air column necessary to produce an effect of resonance is always $\frac{1}{4}$ of the wave length of the sound of the fork. A little analysis of the conditions in the cylinder will enable us to see why this must be so.

One prong of the vibrating fork is shown in Fig. 52. A complete wave consisting of one compression followed by one rarefaction is given off by the fork in moving from *b* to *a* and back again to *b*. Now as the prong starts down from *b* it produces a compression which travels down the tube, is reflected at the bottom and, if it is to produce resonance, must get back to the fork in time to combine with the next upward moving compression produced by the prong. This compression is produced when the prong starts back from *a* toward *b*. It is therefore clear that the compression must travel to the bottom of the tube and back again while the prong, moving from *b* to *a*, is making one-half a complete vibration and is producing half a wave. The distance to the bottom of the tube must therefore be $\frac{1}{4}$ the length of the wave.

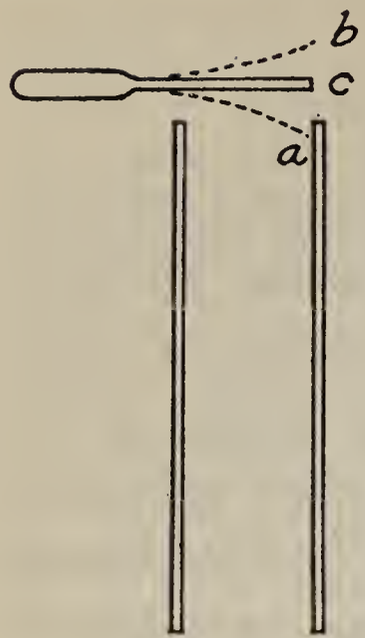

Fig. 53.

Resonance will also be produced if the reflected compression gets back in time to combine with any subsequent upward moving compression, not necessarily the first one. It can be seen from the figure that upward moving compressions are given off after $\frac{1}{2}$ vibration, after $\frac{3}{2}$ vibration, $\frac{5}{2}$, $\frac{7}{2}$ and so on. If therefore the tube is $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$, or any odd number of quarter wave lengths deep, resonance will result.

If, now, instead of using the glass cylinder we use a telescopic tin tube, open at both ends, we shall find that the open tube, if of proper length, will re-enforce the sound of the fork just as the closed tube did, with this difference, however, that the open tube must be twice as long as the closed tube in order to produce the effect. It is found in general that an open tube will produce resonance when its length is one-half the length of one wave of the sound entering it.

In order to understand what goes on in the open tube, a new fact must be brought out. When the downward moving compression made by the fork moving from *b* to *a* (Fig. 53) arrives at the open bottom of the tube, instead of coming against a solid wall and being thrown back, as in the closed tube, it expands out into free air at less pressure than in the tube itself. The particles in the compression therefore " overshoot " and leave behind at the bottom of the tube a space of reduced pressure—a new rarefaction, which will be propagated up the tube just like any other rarefaction. Each compression is

therefore reflected as a rarefaction. This always takes place when a sound wave enters a medium of less density than that of the medium in which it has been moving. We can easily see that in order for resonance to take place, the reflected upward moving rarefaction must get back in exact time to combine with a subsequent upward moving rarefaction just produced by the fork. The first subsequent upward moving rarefaction will be produced behind the fork when it next starts down from *b* toward *a,* that is at the end of one full vibration—succeeding ones will move upward after 2, 3, 4 and any number of full vibrations. The sound must therefore travel down the tube to the bottom and back again to the top while the fork is giving off some whole number of waves. The length of the tube must consequently be a corresponding number of half wave lengths—$\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$ and so on—in order to produce resonance.

4. Beats.—These experiments with the open and the closed tubes show cases of sound wave interference which result in re-enforcing the original sound. An experiment will now be described in which two sets of sound waves interfere in such a way as to diminish or destroy the total sound effect.

Let us set side by side two tuning forks, one with a frequency of 256 vibrations per second, the other with a frequency of 250 vibrations per second. If we sound these two forks together we hear a sound of a distinctly pulsating character, becoming alternately louder and then weaker than the sound of a single fork. This pulsating effect is due to alternate destruction and re-enforcement in the two trains of sound waves. This pulsating is called the "*beating*"

of two sounds and the individual pulses are called "*beats.*"

To explain beating, let us suppose that two compressions, one from each fork, start out together. At the end of one second they will still be together, 1126 feet away from the forks. In the space between the 1126 foot mark and the forks there will be 256 waves from the one fork and 250 waves from the other. In order to examine these waves let us suppose that they remain stationary. At the 1126 foot mark two condensations are together so that there we have a re-enforcing effect. As we move back from the 1126 foot mark toward the forks, we shall find that the waves in the train of shorter waves lose distance as measured against the longer waves. Since there are 256 short waves in the same space as 250 long ones, each short wave must be $\frac{250}{256}$ as long as each long one, and, in each wave length, must lose $\frac{6}{256}$ of a long wave length. It follows from this that after we have gone back $42\frac{2}{3}$ long wave lengths, the short waves will have lost one entire long wave length on the long ones, two condensations will again be together and there will be again re-enforcement. Half-way between, however—$21\frac{1}{3}$ long wave lengths from the 1126 foot mark—the short waves will have lost one-half a long wave length so that at that point a condensation of a long wave and a rarefaction of a short wave will be exactly coincident and there will be a destruction of the sound effect.

It will be noted that the points of re-enforcement being $42\frac{2}{3}$ long wave lengths apart, there must be six of these re-enforcements in the 1126 feet. If now the ob-

Fig. 54.

server remains stationary and the waves move past his ear, he will hear six beats in one second. In general, as this explanation indicates, the number of beats in a second between sounds from two sources is equal to the difference in the frequency numbers of the sources.

The subject of "beating" is of importance because it has been found that discords between sounds are due entirely to this cause. If the number of beats lies between 5 and 50 in each second, the effect on the ear is distinctly unpleasant. The most discordant effect is produced by about 30 beats per second. For instance the notes B and the C next to it, which differ by 32 in their frequency numbers and therefore produce 32 beats each second, are the most discordant on the piano. If the beats exceed about 70 per second, the ear does not detect them and the two sounds seem to blend and produce harmony.

5. Organ Pipes—Open and Closed.—Church organs contain pipes some of which are open and some closed at the upper end. These pipes are arranged so that a plate or jet of air can be blown across a lip at their lower ends. When this jet of air is properly blown so as to strike just inside the lip of a closed pipe, a condensation is produced which goes up the tube—is reflected at the top, just as in our experiment with the glass cylinder, and on its return pushes the jet of air out of the hole in the pipe below the lip. (Fig. 54.) This produces a rarefaction which travels up the pipe and on its return pulls the air jet back in. The jet thus vibrates at a rate determined only by the length of the pipe. This rate is such that the note given by the pipe has a wave length four times that of the pipe.

If the pipe is an open one, it acts in an exactly similar way except that, as pointed out before, condensations will be reflected from the end in the form of rarefactions and the note given will have a wave length twice that of the pipe.

It follows from these statements that if an open pipe be closed with a plug, the wave length of the note that it emits will be doubled, and the second note will therefore be an octave below the first.

In our studies on resonance in a closed cylinder, we found that a tube will resonate to a note if it is $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$ or any odd number of quarter wave lengths deep. This means that a tube of given depth can resonate to any of a whole series of notes of which the lowest or *fundamental* note has a wave length of $4\ l$, if l is the length of the pipe, the next higher a wave length of $\frac{4}{3}\ l$, the next a wave length of $\frac{4}{5}\ l$ and so on. The frequencies or pitches of these notes will clearly be in the ratios 1.3.5.7.9 and so on. The upper notes are called "overtones," being specified as first, second and third overtones and so on in order.

If the pipe is an open one, it will resonate to a note if it is $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$ or any number of half wave lengths deep. A given open pipe will therefore resonate to a series of notes of which the fundamental has a wave length twice the length of the pipe; the first overtone has a wave length equal to that of the pipe; the second overtone has a wave length $\frac{2}{3}$ that of the pipe, and so on. The pitches of these overtones will be in the ratio 1.2.3.4 and so on. Open and closed organ pipes can, when properly blown, be made to give some of these overtones louder than the fundamentals. The blowing

must, however, be violent and does not occur in ordinary playing.

6. Wind Instruments.—Nearly all wind musical instruments operate on the general principles brought out in the last few paragraphs. They can be classified as (1) vibrating air jet instruments, (2) vibrating reed instruments, (3) vibrating lip instruments. The instruments of class (1) are like the pipe organ. In these instruments the rate of vibration of an air jet is controlled by the resonance of pipes of different lengths. The instruments of class (2) may be subdivided into two groups, in the first of which we have the clarinet, oboe, bassoon, etc., while in the second we have the mouth organ and accordion. In the clarinet type there is a reed, which is practically without rigidity, at the mouth end of the instrument. This reed acts under the resonance of the different lengths of pipe opened up by the various stops, just as does the air jet in the organ pipe. In the mouth organ, there is a metallic reed with a definite natural period which is set into motion by the blast of air but is not otherwise affected by it. These instruments do not need any resonating air columns. The instruments of class 3 include the cornet, trombone, bugle, etc. Here the lips act like the flexible reed in the clarinet, the pitch of the note emitted depending on the length of the attached pipe. In the cornet, this length is altered by throwing in or out various links in the tube by means of piston keys. In the trombone, the length is altered by sliding a telescopic tube in and out. In these instruments a number of overtones can be blown for each pipe length by properly adjusting the lips and the the air pressure. In the bugle the length of the pipe

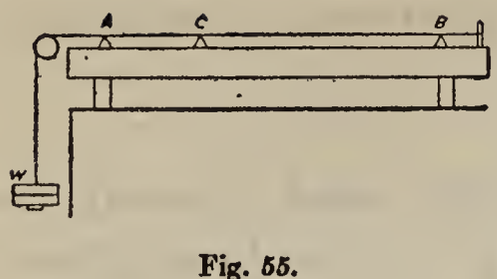

Fig. 55.

cannot be changed so that this instrument can produce only about six notes—its fundamental with perhaps five overtones.

7. Stringed Instruments.—We shall next briefly study the emission of sounds from vibrating strings—strings being the essential parts of all instruments of the type of piano, harp and violin. The factors which affect the pitches of the sounds given by strings can best be studied by the use of the sonometer (Fig. 55). This is merely a sounding box provided with two fixed bridges *A* and *B* at the ends and a movable bridge, *C*, between. The wires or strings under investigation can be stretched across the two fixed bridges and can be subjected to any desired tension by hanging weights *W* on the end as shown. The effective lengths of the strings can be changed by use of the movable bridge.

Let us take two strings of the same material and diameter, subjected to the same tension and adjust the movable bridge so that the length of the one string is half that of the other. If now we pluck these strings, the shorter one will give a note an octave higher than that of the longer one. In other words, it is making twice as many vibrations per second as is the longer string. We would find in general, if we extended this experiment, that other factors being kept constant, *the vibration numbers of strings are inversely proportional to their lengths.* If now we keep the length, materials and diameter constant, and alter the tension, we shall find that to raise the pitch one octave, that is, to double the rate of vibration of the string, we must put on four times the tension. In general, other factors re-

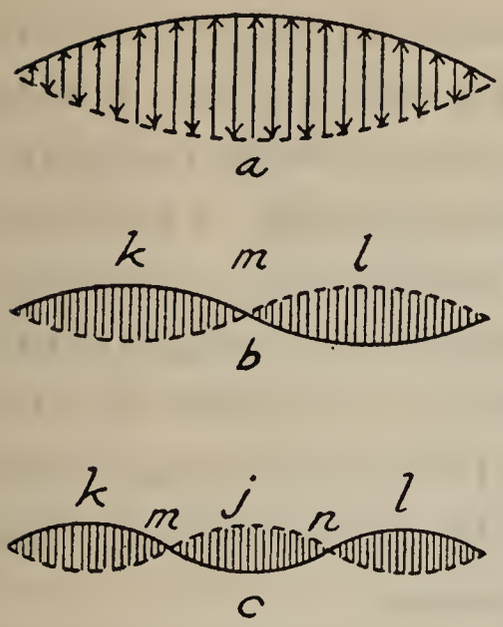

Fig. 56.

maining constant, the *vibration numbers of strings of equal lengths are directly proportional to the square roots of the tensions.*

We shall call these statements the two laws of vibrating strings. The factors of length and tension covered by the laws are the only ones which can be altered in a given string. The pitch also depends on the material of the string and on its diameter.

If we grasp the string of the sonometer exactly at the middle, pull it to one side and then release it, it will vibrate as a whole as shown in *a* Fig. 56. It will then give out the lowest-pitched note of which it is capable; this note is called the *fundamental.* If we touch it at the center and pluck it at one quarter of its length from the end, we can make it vibrate as shown at *b*. Little paper riders, placed at *k* and *l* will be thrown off but a rider put at *m* will remain in place. This stationary point is called a *node.* The points of greatest movement are called *loops.* As we might expect, since the string is here vibrating in two parts, each part half the original length, the pitch of the sound given out is the octave of the fundamental. This is the *first overtone.* If now we touch the string at one-third its length from the end and pluck it at one-sixth its length from the end, we may, with skill, be able to get the form of vibration shown at *c* Fig. 56. Paper riders will remain at *m* and *n* but nowhere else. The sound emitted is the *second overtone.* Its vibration number is three times that of the fundamental.

A string can, similarly, be made to vibrate in any number of equal parts and can therefore give a fundamental note and a series of overtones with frequency ratios of 1,2,3,4,5, etc., to the fundamental. Overtones which are multiples of the fundamental are called *harmonics.* Even harmonics comprise the 2nd, 4th, 6th, 8th, etc., overtones. Odd harmonics comprise the 1st, 3rd, 5th, 7th, etc. It will be noted that strings and open pipes give all the harmonics, odd and even, but that closed pipes give only odd harmonics.

When a string is plucked at random, it ordinarily will give a complex note, including its fundamental and a number of harmonics. The harmonics which have nodes at the point where the string is plucked, are of course absent, since the point plucked cannot possibly be at rest. Since the lower overtones of strings are ones which blend very smoothly with the fundamental, a plucked string gives a pleasing sound. The seventh overtone is the first one that produces unpleasant effects. This overtone is suppressed in the piano by arranging the hammers so that they strike the strings about one seventh of the length of the string from one end.

Stringed musical instruments can be classified into three groups: Group I, those in which the string is struck, as in the piano. Group II, those in which the string is plucked as in the guitar, mandolin, harp, etc. Group III, those in which the string is rubbed or bowed as in the violin, viola, violincello, etc. Since a skilled player can, by adjusting his method of bowing, alter the relative prominence of the overtones in sounds of the same pitch given by instruments of Group III, these instruments are the most expressive of all.

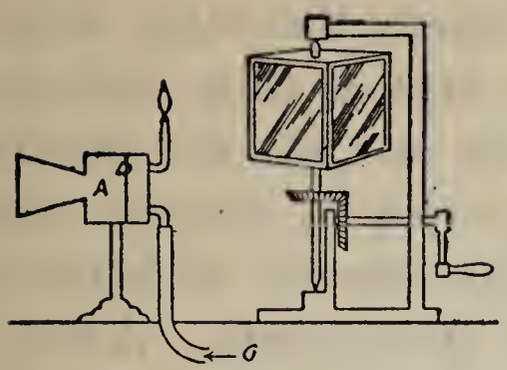

Fig. 57.

8. Quality and Overtones.—We know that sounds of the same pitch and intensity, if arising from different instruments, usually sound different to the ear—the difference being one of what we call *quality*. Experiments have shown that the *quality* of a sound of definite pitch is dependent on the *nature and relative prominence of the overtones present* in it. If the sound has few or weak overtones, such a sound for instance as we get from a gently blown organ pipe, the quality is soft and mellow. If the lower overtones are present up to the sixth, the quality is full and rich. If high overtones, the seventh and above, are prominent, the quality is harsh and metallic or tinlike. The tuning fork when struck with a soft mallet gives a very pure fundamental tone without harmonic overtones. It is because of this purity of tone that tuning forks are so widely used in scientific and musical work.

The ordinary sounds of the human voice or of musical instruments are, then, of complex structure, the actual wave in the air being the resultant obtained by adding to the fundamental sound a number of overtones. If several sounds are passing thru the air at the same time, the wave in the air will be the resultant obtained by adding together all the component waves. These compound waves may be analyzed into their components in any one of a number of ways. A simple device for showing the compound nature of most sounds is the manometric flame (Fig. 57). In this a trumpet communicates with one compartment of a cylindrical box divided into two compartments by means of a stretched rubber diaphragm *D*. Illuminat-

Fig. 58.

ing gas enters the other compartment thru a tube and escapes thru a jet as shown. When the jet is lighted, the flame burns steadily as long as the pressure in the compartment *A* is steady. A condensation entering the trumpet increases the pressure in *A* and causes the flame to rise. A rarefaction entering will on the other hand cause the flame to drop. If the image of the flame is reflected from a cubical rotating mirror successive pictures of it will be obtained which, resembling Fig. 58, will serve to indicate the successive pulses of the compound wave. This device really shows the form of the resultant sound wave and does not actually analyze it into its components.

The various overtones present in a compound wave can be picked out by the use of a set of "resonators" which are vessels of such graduated sizes that each one of them resonates strongly to notes of a single pitch. When a compound sound falls on a set of such resonators, each strong overtone will set its particular resonator into vibration. This vibration can be detected by means of manometric flames attached to each resonator. This method of analysis was developed by the German scientist von Helmholtz (1821-1894). The resonators (Fig. 59) are frequently called Helmholtz's resonators. After an analysis has been made by this method, the quality of the original sound can be fairly well reproduced by blending the notes of a number of tuning forks, giving the same pitches as the resonators excited in the analysis.

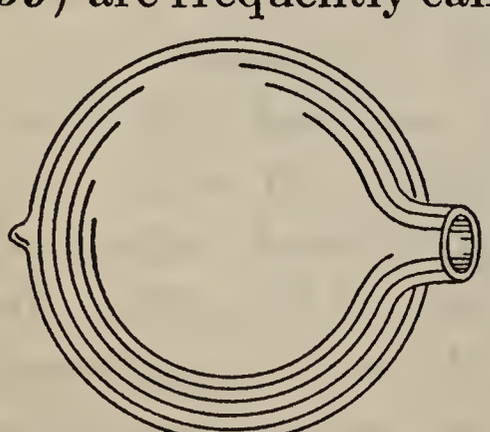

Fig. 59.

9. The Phonograph.—The phonograph is an instrument for recording and reproducing sounds. In making the original record the sound is allowed to fall on a flexible diaphragm provided, on the under side, with a sharp needle-like point. This point runs in a groove on a soft wax or composition sheet which is being rotated by clock work. The successive compressions and rarefactions of the sound wave produce corresponding movements of the diaphragm and its needle, which result in the formation of indentations of varying shape and depth in the groove on the composition sheet. If this disk, after hardening, is rotated at the same speed as before, under another diaphragm provided with a needle of somewhat different shape, which runs in the indentations in the grooves, the second diaphragm will reproduce the movements of the first with a fair degree of accuracy and a sound similar to the original sound will be given off. There are many different types of phonographs, but all work on this general principle.

REVIEW.

1. What are the pitch ratios of the Major Diatonic Scale?
2. Explain why the wave length of a note given by an open organ pipe is twice that of the pipe, while in a closed pipe the wave length is four times the length of the pipe.
3. Explain the production of overtones in organ pipes.
4. Is there sound when there is no ear to hear it?

CHAPTER X

LIGHT—ITS NATURE AND PROPAGATION

1. Nature of Light.— Our knowledge of the external world is gained entirely from the impressions received from our various senses. A reliable impression comes from the sense of touch when some part of our body is in direct contact with the object observed. Equally important, altho frequently subject to strange illusions, are the impressions obtained thru the sense of sight. In the latter case there is no obvious connection between the observer and the object observed. Since placing the hand over the eyes cuts off all sight sensations, it is evident that the eyes are the organs of sight.

The problem of what it is that, passing between the object and the eye, renders the object visible has occupied the attention of philosophers for many centuries. The ancients supposed that something, in the nature perhaps of immaterial tentacles, extended from the eye to the object, but it seems more logical to suppose that this something emanates from the object and enters the eye, there stimulating the optic nerve. This something we call " light." The absence of any stimulation of the optic nerve such as we have in a completely closed black room indicates the complete absence of light, which we specify as " darkness."

We know that some objects, such as flames, phosphorus or the incandescent filaments of electric lamps, can be seen in a black room. Such objects, in order to be seen, must emit light, and are therefore called self-

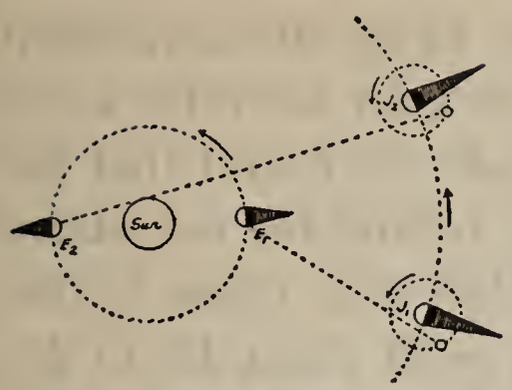

Fig. 60.

luminous bodies. We know that all objects which are not self-luminous can be seen only when light from some luminous source falls on them. For instance, in the day time we see the furniture in the room by the direct or indirect light from the sun; at night we see it only when light from some lamp or similar source is falling on it. The light emitted from the self-luminous source is evidently reflected from the non-luminous bodies and thence, passing to the eye, produces the sensation of sight.

2. Propagation of Light.—Without at present saying anything more definite about the nature of light, a number of important facts can be brought out touching on the way in which light is transmitted or propagated from a source. In the first place it is evident that if light emanates from a source and travels to the eye, it must travel at some definite measurable rate. This rate must be very high since there is no perceptible delay in the receipt of such sensations as the flash of a distant gun, altho there is a very long delay in the receipt of the associated sound. The rate of speed is, as a matter of fact, so high that for many centuries it was supposed that light was instantaneously transmitted.

The first measurement of the rate was made in 1675 by Roemer, a Danish astronomer. Roemer had accurately calculated the rate of revolution of one of the moons of Jupiter so that he could foretell the exact instant in the future at which it would next enter the shadow of Jupiter and thereby become eclipsed. He made these calculations for the distance from Jupiter to

the earth when the earth was at E_1 (Fig. 60) in its orbit and Jupiter was at J_1. Six months later when the earth had moved to E_2 and Jupiter to J_2, he found that the eclipse was 16 minutes 36 seconds behind the calculated time. He ascribed this delay to the time taken by the light to travel across the earth's orbit from E_1 to E_2. Now this distance was known to be 186,000,000 miles whence we can easily calculate that the light must have traveled about 186,000 miles in each second. This is equivalent to about 300,000 kilometers per second. Subsequent measurements have confirmed this value—the best determination giving 299,860 kilometers per second. This enormous speed is sufficient to carry light around the equator of the earth $7\frac{1}{2}$ times in one second.

It is interesting to note in this connection that the fixed stars are so distant that it takes 4.4 years for the light from the nearest one to get to us. The light from the Polar Star takes 45 years to travel to the earth while many of the stars are several hundreds of light years away from us.

The rate above given is the rate at which light travels in the space between the earth and the sun where there exists a very perfect vacuum. It has been found that when the light enters any other medium, such as air or water, its rate is reduced. The reduction on entering air is very slight, being in the ratio of 1 to 1.003, so that in our work we can neglect any consideration of this reduction. In water, however, the velocity is only about $\frac{3}{4}$ of what it is in free space. These changes in velocity have important consequences as we shall see later.

3. Reflection.—If we allow a beam of light from the sun to enter a darkened room thru a small hole in the window shutter, we can trace the beam by the illuminated dust particles in its path. The path is seen to be a straight line. If we hold a screen in the path the beam is completely cut off at that point. If the screen is of white rough paper, light will be reflected or diffused from it into all parts of the room rendering objects in the room visible, but if the screen is a polished metal mirror, the direction of the beam will be altered by reflection without any spreading out. The difference in the character of the reflection in the two cases is due to the difference in the character of the two surfaces. The rough surface is made up of a great many small surfaces, all at different angles with one another, from which the beam is reflected, so that some light is thrown off in every direction. No matter at what angle we look at the rough surface we can see it, because some light is certain to be diffused in the direction of our eyes. When, on the other hand, the beam strikes the smooth metal mirror, it meets a surface all parts of which are in one plane, so that the whole beam is reflected in one direction. If we look at the surface from any other direction it will be, if absolutely clean, entirely invisible since there is no light reflected in the other directions.

If we look along the reflected beam into the mirror we see, since the beam is unaltered by the reflection, not the surface of the mirror but the source of the light, just as if we had looked along the original beam before it was reflected. This brings out the fact that we cannot see the shape and surface of a body unless it reflects light diffusely. If a body reflects light regularly like a mirror or not at all like a perfectly transparent body or like an

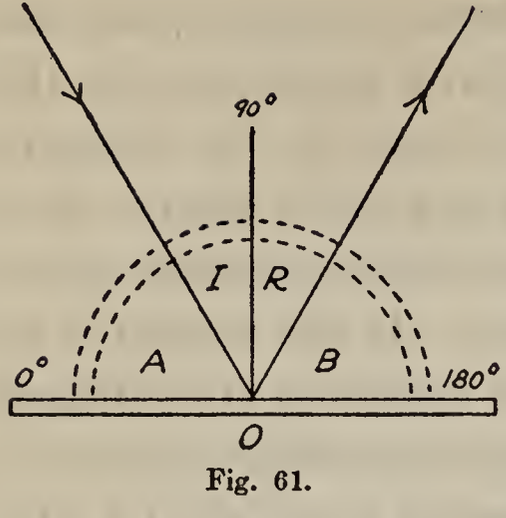

Fig. 61.

absolutely black body, then we cannot see it except as it stands out against a background of diffusely reflecting materials. This fact is made use of in many tricks of the stage magician and is also illustrated by the difficulty of perceiving the very large mirrors which are frequently set at the ends of stores to increase their apparent size.

If we set an angular scale beside the mirror so that we can compare the angles which the incident beam of light and the reflected beam form with the face of the mirror, we at once discover that a very simple relation exists between these angles. If the mirror is flat, the angles *A* and *B*, Fig. 61, are found to be in all cases equal. If we erect a perpendicular, or *normal* as it is frequently called, at *O*, the angles *I* and *R*, called respectively the angles of incidence and of reflection, are also equal. The *law of regular reflection* is usually expressed by saying that the *angle of incidence equals the angle of reflection and lies in the same plane.* Now the angle of incidence is the angle between the incident ray and the normal to the surface at the point of incidence, and the angle of reflection is the angle between the reflected ray and the normal to the surface at the point of incidence. To say that these angles are in the same plane means that the incident ray, the reflected ray and the normal must lie in the same plane. This law is true in the case of a single ray of light whether the mirror is flat or not.

4. Shadows.—Since light travels in straight lines, it follows that if a screen which does not transmit light—that

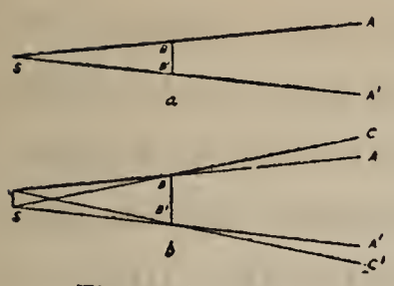

Fig. 62 (a and b).

is an opaque screen—is introduced into a beam of light, none of the light can get into the space behind the screen. This space which, if not illuminated from other sources, will be in complete darkness, is called a "shadow." Let us study briefly the character of the shadows produced by objects when illuminated by sources of different sizes.

First, suppose the source is very small—a "point source." Let the opaque object be a circular disk larger than the source. The conditions in this case are shown in Fig. 62a. The light is completely excluded from the space A^1B^1BA. This space of complete darkness is called the *umbra* of the shadow. In this case the shadow is all umbra, and is in shape a truncated cone.

Next, let the source be a disc of finite size but smaller than the opaque disc BB^1. These conditions are shown in Fig. 62b. The light is completely excluded from the space A^1B^1BA which, being in complete darkness, constitutes the umbra, in shape a truncated cone as before. Outside the umbra, however, there is a region ABC-$A^1B^1C^1$ from which part of the source is visible. At A only the beginning of one edge of the source can be seen while at C the entire source can be seen. The intensity of the shadow therefore becomes less in passing out from A until it entirely disappears at C. This region of partial illumination is known as the *penumbra*.

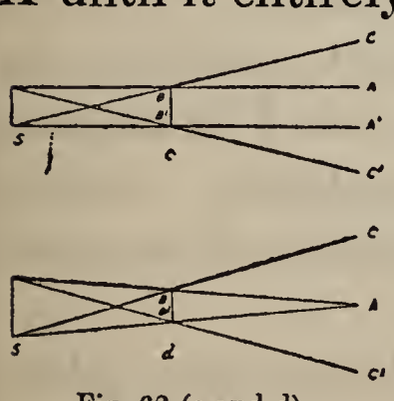

Fig. 62 (c and d).

Next let source and screen be discs of the same size as Fig. 62c. The shapes of umbra and penumbra in this case are clearly shown in the figure.

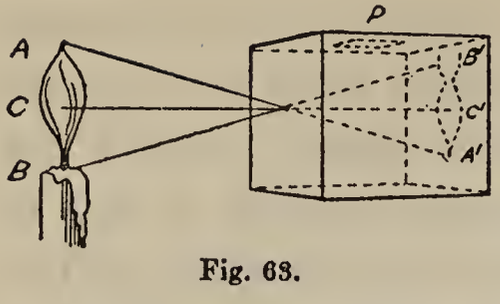

Fig. 63.

Finally, let the source be larger than the screen as in Fig. 62d. Here the umbra is a cone around and beyond which the penumbra, relatively much larger, extends. The shadow thrown by the earth in the light of the sun is an illustration of this last case. The moon is a cold sphere which we see by the light of the sun reflected from it. When the moon, in revolving around the earth, passes into the earth's shadow we can no longer see it and it is said to be "eclipsed." The distance of the moon from the earth is such that it is near the conical point of the umbra and therefore frequently misses the shadow on its way around. If this were not so the moon would be eclipsed every 28 days.

Shadows and eclipses offer excellent illustrations of the straight line (or rectilinear) propagation of light. Another illustration is afforded by the device known as the pinhole camera. This is a light tight box in the front wall of which is a pin hole (Fig. 63). If a candle is placed in front of the hole, the screen at the back of the box will be illuminated by the candle—but as a study of the figure will show, A^1 will be illuminated only by light from the tip of the flame, and B^1 only by light from the base, and so all points on the source will illuminate only a corresponding small area on the screen. This illuminated area will therefore have the shape, color and movement of the source, and will in fact constitute an "image" of it. If we look in thru the hole P in the top of the box, we can see this image very plainly. As the illustration shows, the image will be upside down with reference to the object.

If in place of the screen we put a photographic plate

at the back of the box, a picture can be taken of the image, but the process will be slow on account of the small amount of light that can enter the pin hole. The image will be formed on the screen no matter what may be the latter's distance from the hole, but the greater the distance the larger the image will be and the more faint, since the same amount of light is spread over a greater surface.

5. The Wave Theory of Light.— We have now found out that light, traveling with a speed of 186,000 miles per second, is emitted from a luminous source in all directions in straight lines. We know that it is reflected by certain surfaces either diffusely or regularly. In the history of modern science there have been two principal theories as to the nature of light. The first of these theories was known as the " corpuscular theory." Light was supposed to consist of streams of extremely minute material particles, called " corpuscles," which were emitted by luminous bodies and traveled out at the rate of 186,000 miles per second in straight lines. This theory is capable of explaining most of the phenomena of light except those of interference and diffraction of which we shall speak later. The second theory was the " wave theory " of light which was first completely expressed by the Dutch physicist, Huygens (1629-1695). Under this theory light is supposed to be a wave motion, similar to that of sound, the waves being given off from luminous bodies. In order to explain the transmission of these waves thru empty space, Huygens assumed all space to be filled with a weightless elastic material called " ether " which had the property of transmitting light waves and is therefore frequently called the " luminiferous ether." This theory was examined and rejected by

the famous Sir Isaac Newton because at first it did not seem to account for the straight line propagation of light. This difficulty was subsequently overcome, and since about 1800, when it was found that no other theory could offer a satisfactory explanation of the "interference" of light, the wave theory has become more and more firmly established until at present it is universally accepted as true.

6. Phenomenon of Interference.— Since the phenomenon of interference was the one which finally determined the adoption of the wave theory, it is worth while to consider an experiment illustrating it. Two long narrow strips of plate glass are clamped together with a sheet of paper between them at one end. This leaves a very flat wedge of air between the two plates. An alcohol or non-luminous gas flame, such as is given by a Welsbach burner without a mantle, is now touched with a piece of paper moistened with common salt solution. The salt volatilizes in the flame and gives it an intense yellow color. When we view the reflection of this flame in the glass plates, we see that the surface is crossed by a long series of distinct yellow and black lines. The formation of these lines can be explained on the basis of the wave theory as due to the interference of light waves, but cannot be explained at all on the basis of the corpuscular theory. When the yellow light strikes the glass it is reflected from all four surfaces but in our explanation we shall consider only the reflection from the back surface of the front glass and the front surface of the back glass, that is to say, from the two surfaces bounding the wedge of air between the plates.

Let AB and CD (Fig. 64) represent these faces of the plates. Let the full lines represent waves reflected

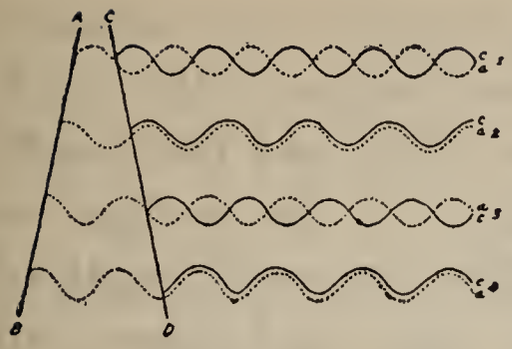

Fig. 64.

from *CD* and the dotted lines waves reflected from *AB*. These waves are drawn in to represent light waves with crests and troughs such as can be produced in a rope by shaking it. The motion of the particles in this type of wave are at right angles to the line in which the wave is moving, whence such waves are called *transverse waves.* It is well to remember at this point that in sound waves the particles move back and forth along the line of motion of the wave, whence sound waves are called *longitudinal waves.* Altho light and sound are both wave motions, they are seen to be composed of different types of waves.

Referring again to the figure; along line 1 the wave *c* is seen to meet the wave *a* in such a way that at every point a crest on one wave is opposed by an equal trough on the other. The result of combining the two waves in this way will be to destroy both. On looking at the glass along 1 we therefore see a black line. Along 2, on the other hand, the crest of wave *a* combines with the crest of wave *c,* producing a re-enforced wave which will appear a bright line to the eye. At 3 we find again destruction and at 4 re-enforcement and so on, thus accounting for the observed black and yellow lines.

Assuming "one wave" to be built up of one crest and the next following trough, it is easy to see that if destruction takes place along any line as 1, it must take place again along another line, as 3, where the path of the light thru the air wedge and back again is exactly one wave length longer than at 1, because then the conditions of interference at 3 will exactly reproduce those

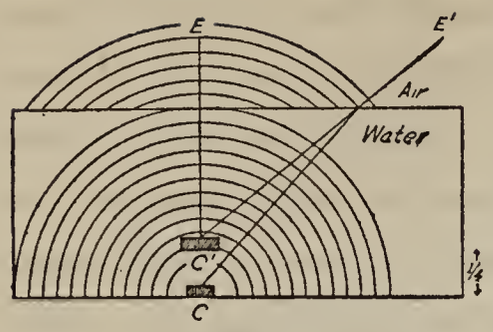

Fig. 65.

at 1. If, therefore, we measure the distance between the black and yellow lines and the length and thickness of the wedge, we can calculate the wave length of the yellow light and figure out the rate of vibration of the particles in the flame which give out waves of this length. If this were carefully done, it would be found that each wave of the yellow light is only .00059 millimeters in length. Since light travels at the rate of 300,000 kilometers per second, the number of vibrations per second corresponding to this wave length ($l = \frac{V}{n}$, just as with sound) is enormous, being about 500,000,000,000,000 per second.

7. **Refraction.**—We may now apply this wave theory to the explanation of a few phenomena. Take the case of a coin lying at the bottom of a vessel of water as shown in Fig. 65. The light reflected from the coin by which we see it, spreads out in spherical waves concentric on the coin. When these waves leave the water, they immediately begin to move at a rate $\frac{4}{3}$ as fast as before. This results in a bulging up of the wave with an increased curvature as shown in the figure. When this wave enters the eye at E or E^1, it will appear to the eye as if the object were at C^1 rather than at C the true position. The vessel of water therefore appears more shallow than it really is.

If instead of considering the whole wave we take a very small portion of the wave front—or, as we shall call it, a " ray "—the conditions will be as shown in Fig. 66. It will be noticed that on emerging the ray coming

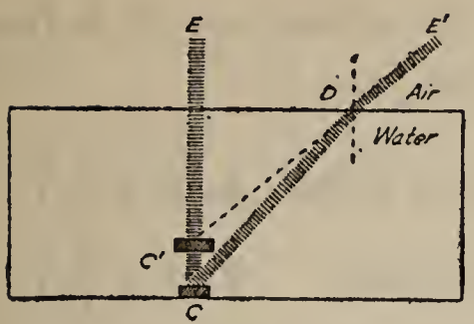

Fig. 66.

up to the surface obliquely is bent away from the perpendicular to the surface, while the ray coming straight out normal to the surface is not bent at all. It is evident that a change in the curvature of a light wave must take place every time that it enters or leaves a medium in which its speed is different from the speed it had before. A ray of light will consequently always be bent to one side or the other on entering or leaving such a medium, unless it enters or leaves on a perpendicular to the surface. This bending of light rays on passing from one medium to another is called *refraction.*

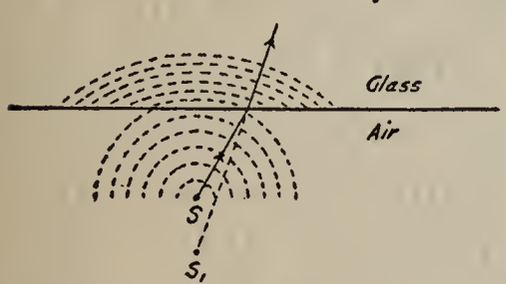

Fig 67.

The conditions when the light passes from a medium where its speed is higher to one where its speed is lower are shown in Figs. 67 and 68, where the upper medium is taken to be glass and the lower air. It will be noted that the ray of light on entering the glass is bent *toward* the perpendicular to the surface. The amount of bending is seen to depend on the change which takes place in the curvature of the wave which in turn depends on the relation of the speeds in the two media. Knowing the relation of the speeds, it is possible to figure out the direction and amount of the bending. These speed relations are usually expressed with reference to the speed in air. The ratio of the speed in air to the speed in any other medium is called the *index of refraction* of that medium.

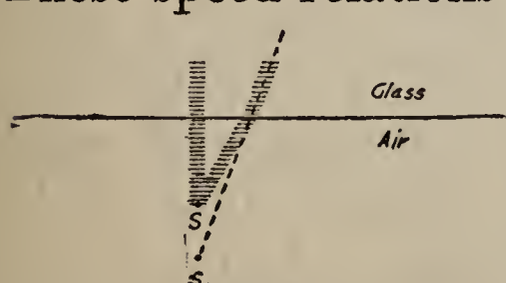

Fig. 68.

The indices (plural of index) of refraction of a few substances are shown in the following table:

Water	1.33 $(\frac{4}{3})$	Crown Glass	1.53 $(\frac{3}{2})$
Flint Glass	1.67 $(\frac{8}{5})$	Diamond	2.47 $(\frac{5}{2})$

In studying the refraction that takes place in any particular case, it is more convenient to deal with beams of light, that is, with small areas of wave fronts, rather than with whole waves. Let us note the refraction which occurs when a beam of light passes thru a block or plate of glass with flat parallel sides. (Fig. 69.) The beam comes down along the line AB. When it strikes the glass, the wave fronts are retarded on the edge that enters first, so that the beam is bent into some direction BC, toward the perpendicular NN_1. The amount of this bending depends on the amount of the retardation and is consequently greater the higher the index of refraction of the glass. When the beam arrives at C, the edge of the wave emerging first will be accelerated by the same amount that it was previously retarded, and the emergent ray CD will swing round parallel to AB but displaced to one side as shown. The bending on emergence is away from the perpendicular N_2N_3.

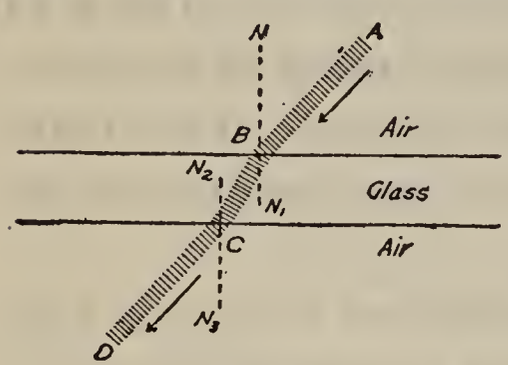

Fig. 69.

Next take the case of a glass prism, Fig. 70. Here again the beam of light is bent *toward* the perpendicular NN_1 when entering, and *away* from the perpendicular N_2N_3 when emerging. Again, as always, the direction of the bend-

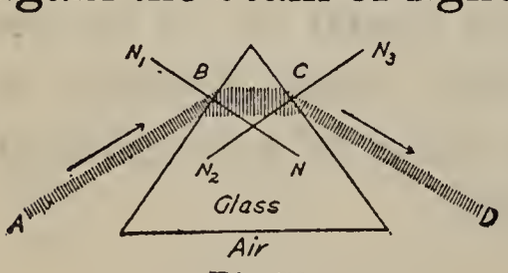

Fig. 70.

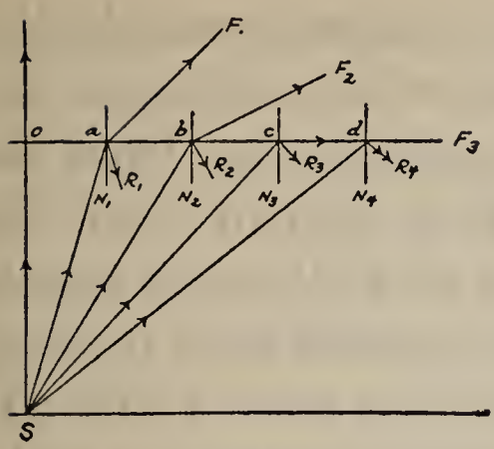

Fig. 71.

ing can be foretold by noting whether the edge of the beam which first enters the new medium is retarded or accelerated. It will be noted that a beam when passing thru a prism of the sort shown here must always be bent around the base of the prism.

8. The Critical Angle.—Lastly let us consider the more complicated case of light coming up from a source at the bottom of a pond of water. (Fig. 71.) Here it will be necessary to take into account the reflection of the light from the under side of the water surface. It must be understood that whenever light enters a medium in which its speed is changed, it is—just as is the case with sound—always in part reflected in accordance with the law of regular reflection which we have already considered. The part of the ray which is not reflected passes on and is refracted. Whenever we look obliquely thru a window pane, we can always see reflections of objects on our side of the window as well as the objects on the other side of the window, seen thru the glass. Bearing this fact in mind, it will be seen (Fig. 71) that the ray passing from s to a is partly reflected along R_1—the rest of the ray passing thru the surface is bent along the line aF_1. The ray coming up to b is also partly reflected and partly refracted. The refracted part bF_2 will be bent more than was the ray aF_1. That this must be so is evident from a consideration of the emerging wave front of the beam which, coming up more obliquely, will have its lower edge longer in the retarding medium after the upper edge enters the accelerating medium, and will

thus swing farther around before emerging completely. Therefore, the greater the obliquity of the incidence, the greater will be the amount of bending away from the perpendicular on emergence. It is evident from the figure that an angle of incidence as ScN_3 must finally be reached for which the refracted ray will be so far bent that it will pass along the surface as it does at cF_3. If the angle of incidence is still further increased to SdN_4 no refracted ray can get out into the air. The entire beam is then reflected along dR_4. This is known as *total reflection.* The angle of incidence ScN_3 at which the refracted ray passes along the surface without actually emerging is known as the *critical angle.* Whenever a beam of light enters a medium in which its speed is increased, it will be totally reflected if the angle of its incidence on the second medium is greater than the critical angle. It will be remembered that the angle of incidence is the angle between the incident ray and the normal to the surface at the point of incidence. The critical angle for water is 48.5°, for crown glass it is 42.5°, and for diamond it is 23.7°. If an eye were at S it would, on looking along Sb, see the sky; on looking along Sc the shore, and on looking along Sd a reflection of the bottom of the pond. When swimming in clear still water, it is easy to verify these conclusions. Since, when a beam of light enters a medium in which its speed is *reduced* it is bent *toward* the perpendicular, it is evident that under these conditions, total reflection can never result.

9. Intensity and Distance.—We shall next consider the variation in intensity of illumination with distance from the source of light. Here, as with sound waves, we shall define intensity as the amount of energy on each unit of

surface, say, on each square centimeter of one of the concentric spherical wave fronts. Now it is evident that exactly the same reasoning applies here as applied in the case of the sound waves. The total energy on one of the expanding spherical wave fronts remains constantly the same. The amount per unit area, the intensity of illumination, must therefore vary inversely with the change in area of the wave front. Since the areas of spheres vary with the squares of the radii, the intensity varies inversely with the square of the radius of the wave front—that is inversely with the square of the distance of the point at which the intensity is measured from the source. This is again the Law of Inverse Squares. The amount of light energy falling on a unit area of a screen, that is the intensity of illumination on the screen, will be reduced to one-fourth, if the distance of the screen from the source is doubled and so on.

It is of course plain that the illumination on a surface varies not only with the distance of the surface from the source but also with the amount of light given out by the source in a unit time. The total amount of light emitted by a source in unit time is called the *illuminating power* of the source. The unit of illuminating power is the light given by a so-called *standard candle* which is a sperm candle $\frac{7}{8}$ of an inch in diameter burning at the rate of 120 grains or 7.78 grams per hour. The illumination produced on a surface by a source at a given distance must naturally vary directly with the illuminating power of the source. If we represent the illumination on the surface by I, the distance from source to surface by d and the illuminating power of the source by P, then we have $I \propto P$ and $I \propto \frac{1}{d^2}$—that is, $I \propto \frac{P}{d_2}$.

Since $I \propto \frac{P}{d_2}$ then $I_1 : I_2 :: \frac{P_1}{{d_1}^2} : \frac{P_2}{{d_2}^2}$ and, multiplying means and extremes, $\frac{P_1}{{d_1}^2} I_2 = \frac{P_2}{{d_2}^2} I_1$. Now if two sources are placed at such distances d_1 and d_2 from a surface that they appear to illuminate it equally, then $I_1 = I_2$ and, cancelling these equals out, our equation reads $\frac{P_1}{{d_1}^2} = \frac{P_2}{{d_2}^2}$ or $\frac{P_1}{P_2} = \frac{{d_1}^2}{{d_2}^2}$. That is to say: the candle-powers of two sources of light are directly proportional to the squares of their respective distances from a surface which they illuminate equally.

This last relation is made use of in important practical measurements of the illuminating powers of sources of light.

10. The Photometer.—The *photometer* is an instrument making use of this principle for comparing the candle-powers of two sources. There are many different types of photometers making use of different devices for determining when the test surface is equally illuminated by both sources. The most common type is the Bunsen or "grease-spot" photometer (Fig. 72). The screen S in this instrument is a piece of paper with a grease spot in the center of it. As any one can easily find out by making a test, a grease spot on a piece of paper is distinctly visible only when unequally illuminated on the two sides. If the illumination on the far side is the stronger the spot appears darker. When the illumination is equal on both sides it is very difficult to see the spot from either side. This screen is placed on a carrier moving on a graduated scale near the middle. At one end of the scale a standard candle C is placed on a similar carrier while at the other end is placed the lamp L, the candlepower of which is to

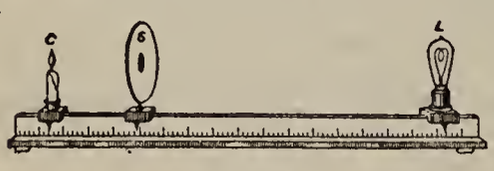

Fig. 72.

be measured. The lamps and screen are moved until the grease spot becomes invisible or nearly so. Then if d_x is the distance of the lamp being measured and d_s the distance of the candle from the screen $P_s : P_x :: d_s^2 : d_x^2$. Since $P_s =$ one, $P_x = \frac{d_x^2}{d_s^2}$.

The candle-power of an ordinary open gas flame burning 5 cubic feet of gas per hour is 16 to 20, depending on the kind of gas used. A Welsbach lamp with a mantel burns about 3 cubic feet per hour, and has a candle-power ranging between 50 and 100. The ordinary arc lamp has a candle-power of only about 500, but its light is not equally distributed, so that in the areas of greatest illumination the effect is equivalent to that of 1000 or even 1200 candles.

REVIEW.

1. What is the rate of speed at which light travels thru space?
2. Explain the wave theory of light.
3. Distinguish between angles of incidence and of reflection.

CHAPTER XI

THE FORMATION OF IMAGES AND OPTICAL INSTRUMENTS

1. Virtual and Real Images.—Our work on light has so far been concerned mainly with the nature and propagation of light. We shall now turn our attention to the important subject of *image formation.*

Whenever an object appears to be in a position other than that which it really occupies, the false or seeming object is called an *image.* We are all familiar with the images seen in ordinary mirrors. An object viewed in a plane mirror appears to be behind the mirror. Since the mirror itself is opaque it is clear that no rays of light from the object can possibly pass thru the points where the image appears to be. Such an image is called a *virtual image.* A virtual image cannot be thrown on a screen placed in the position where the image appears to be or elsewhere. If an ordinary burning glass is held in sunlight, an image of the sun is formed a little distance from the glass on the side away from the sun and this image can be thrown on a paper screen. The rays of light from the sun actually pass thru this image as can be perceived by allowing the image to fall on the tip of the finger. Such an image is known as a *real image.* Real images can always be thrown on a screen, placed in the proper position. It is important to distinguish clearly between real and virtual images.

The formation of images depends primarily on the tendency of the eye to see objects along the line on which the light rays from the object actually enter the eye.

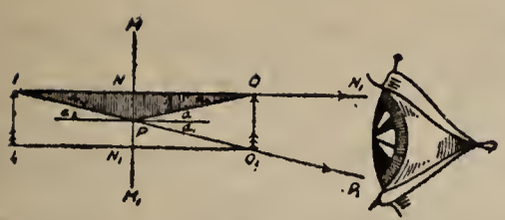

Fig. 73.

We have already referred to this tendency in connection with refraction and reflection in a vessel of water. It is evident therefore that images can be formed only when the rays of light from the object are changed in direction by reflection or refraction between the object and the eye. The object then appears to be in a position where it actually is not; consequently we have an image.

2. Image Formation by Reflection.—We shall study first the formation of images by reflection. In explaining the mode of formation of the various images, we shall use instead of the spherical wave fronts, which when drawn on paper are confusing to the eye, restricted portions of the wave fronts or "rays" of light coming out from the object. These rays when reflected from a mirror will follow, of course, the general law of reflection according to which the angle of incidence equals the angle of reflection and lies in the same plane with it.

Take first the case of reflection from a plane, in other words a flat, mirror. In Fig. 73 let MM_1 represent the mirror and OO_1 the object. The ray of light coming out from O along the line ON will be reflected back on itself along the line NN_1 because the angle of incidence being zero°, the angle of reflection must also be zero°. Another ray as OP meeting the mirror obliquely will be reflected so that the angle a equals the angle a_1, that is along the line PP_1. If now both of these rays, NN_1 and PP_1, enter the eye, the eye will see an image of the point of the object along both these lines, that is, where the lines intersect at I behind the mirror. It can be seen from the figure that angle a equals a_2, equals angle a_3,

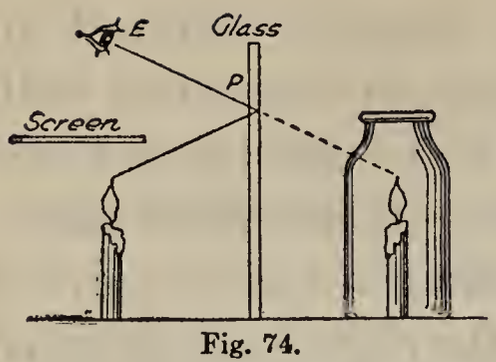

Fig. 74.

equals angle a_4, whence the two triangles *PNI* and *PNO*, shown shaded in the figure, are equal and *NI* must equal *NO*. Similar constructions could be made for all other points on the object OO_1. If so made, the image will be found to be along the line II_1.

The general simple rule brought out by our construction is that the image of any point seen in a plane mirror is on the *perpendicular from the point to the mirror and as far back of the mirror as the object is in front of it.* When this rule is applied to find the image of the lower end of the object O_1 and thus outline the whole object, it is only necessary to draw the line $O_1N_1I_1$ perpendicular to MM_1 and to measure off N_1I_1 equal to O_1N_1. It can be seen from this that the image of an object in a plane mirror must be the same size as the object; it must be erect, that is, with the same end up as the object, and as far back of the mirror as the object is in front of it. The image is of course always *virtual.* If we set up a sheet of clear plate glass as shown in Fig. 74, place a bottle behind the glass and a candle as far in front of the glass as the bottle is behind it, then if we look down along the line *EP*, the candle will appear to be burning inside the bottle. This arrangement or a similar one is frequently made use of in producing illusions on the stage. A stage ghost, for instance, can be made to appear among the actors by causing the audience to view the actors thru a large sheet of clear plate glass, the edges of which are concealed. The "ghost" is actually off stage at the side—his image of course appearing along a perpendicular from his body to the

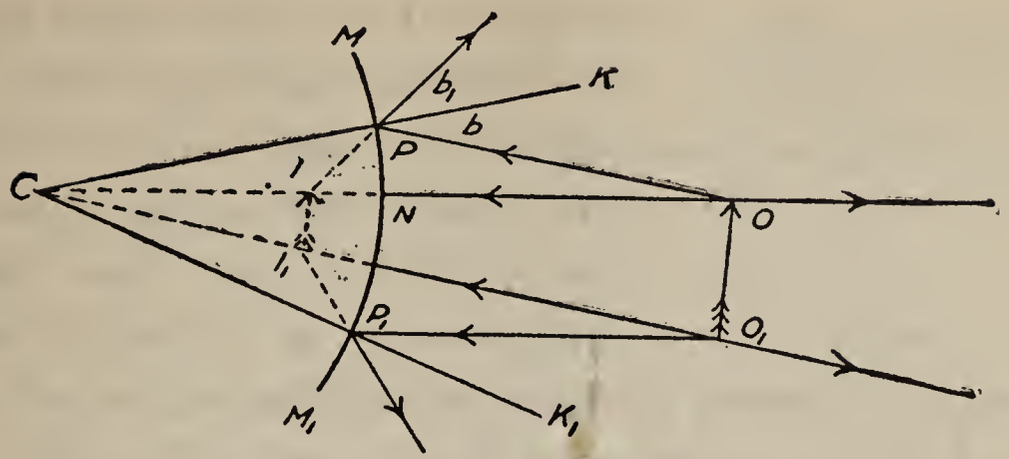

Fig. 75.

mirror and as far behind the mirror as he is in front of it.

3. Convex Spherical Mirror.—Next let us study the formation of images in a convex spherical mirror, that is to say, in a mirror which is a portion of the outside surface of a sphere. Here again we make a construction applying the same law of reflection, only remembering that the normal or perpendicular to this spherical surface at any point is a radius of the sphere drawn outwards from its center. Fig. 75 represents the construction. The ray ON is reflected back upon itself, since it is purposely drawn perpendicular to the surface of the mirror. The ray OP is reflected at P so that the angle b_1 equals the angle b. If these two rays enter an eye, the eye will see the image of the object where the rays intersect, namely, at I. In a similar way, the image of O_1 will appear at I_1. It will be seen that when the object is moved up toward the mirror, the image will also approach the mirror, becoming larger altho it can never become larger than the object. If the object is moved away from the mirror, the image also moves away toward C, becoming smaller. However, no matter how distant the object may be, the image can never get beyond C, the center of curvature of the mirror.

It is also evident from the construction that the image

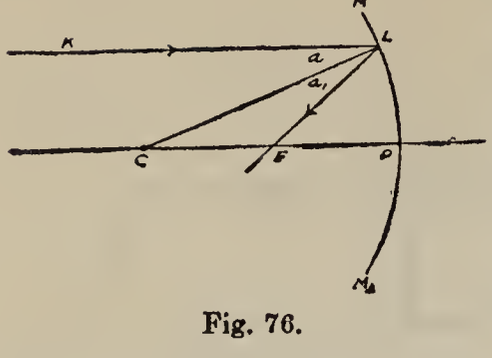

Fig. 76.

will be at all times right side up or erect and always virtual, since there is no light back of the mirror where the image appears to be. We can say then that an image seen in a convex mirror always lies between the mirror and its center of curvature and that it is always virtual, erect and smaller than the object. Mirrors of this type are very extensively used on the windshields and mudguards of automobiles, to give the driver a general view behind him. Such mirrors since they disperse the light which falls on them can never form real images. They are frequently called "dispersing mirrors."

4. Concave Spherical Mirror.—Next we must consider concave spherical mirrors, those which constitute part of the interior surface of a sphere. Altho exactly the same principles as before are used in locating the images produced by this type of mirror, the conditions are somewhat more complicated. Since the concave mirror produces, as we shall see, real images, it is of more practical importance than is the convex mirror. Consider such a mirror as shown in Fig. 76. Any ray striking this mirror will be reflected so that the angle of reflection a_1 with the normal drawn out thru the center of curvature of the mirror at C equals the angle of incidence a. When the incident ray is parallel, as shown at KL, to the so-called axis of the mirror CP, joining the center of curvature with the center of the face of the mirror, then it can be proved that making a_1 equal to a, the reflected ray will cut the axis exactly half-way between C and P. This point F, halfway between C and P, is called the *principal focus* of the mirror.

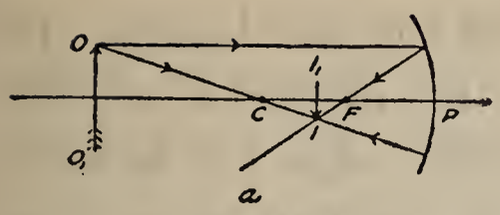

Fig. 77.

Any and all rays drawn parallel to the axis will after reflection pass thru *F*. Since any small section of a spherical wave front from a very distant source is practically flat or plane, all the rays which can be drawn to make up this wave will be parallel to one another. When a plane-faced wave of this sort, such for instance as comes from the sun, falls on a concave mirror, it, after reflection, all passes thru the focus where there will be a very considerable concentration of light and heat. On the other hand, if a source of light is placed at *F*, the light from it that strikes the mirror will after reflection be thrown forward in a parallel beam along the axis. This principle is made use of in cheap reflectors for small lamps.

If we call the fact that any ray drawn parallel to the axis will, after reflection, pass thru the focus, the first principle of image construction in concave mirrors, we can readily derive a second principle by observing that light coming up to the mirror along a radius is reflected back upon itself. We can state this second principle in this way: any ray drawn thru the center of curvature will after reflection be thrown back on itself. The successive application of these two principles will enable us to locate the image of any point on an object in front of a concave mirror, by drawing only two rays, for the image will lie where these rays intersect.

It is necessary however, to distinguish three cases: (1) When the object is outside the center of curvature; (2) when it is between the center of curvature and the focus; (3) when it is between the focus and the face of the mirror. Fig. 77 represents the first case, 78 the second

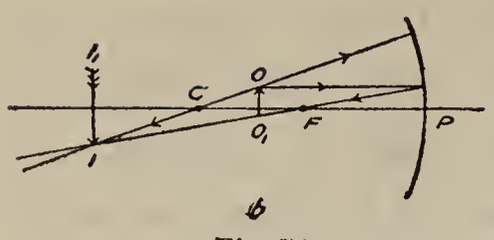

Fig. 78.

case, and 79 the third case. In each case the image I of the point O on the object is located by drawing its two rays, one parallel to the axis passing after reflection thru the focus; the other thru the center of curvature passing after reflection back on itself. The image I_1 of the point O_1 could be located by an exactly similar procedure.

A critical study of these constructions shows that in cases *a* and *b*, the light from the object after reflection actually passes thru the place where the image appears to be, so that these images are both *real.* In both these cases, the images are *inverted,* that is, up-side down, the image in case *a* being *smaller* than the object and between C and F, while in case *b*, it is *larger* than the object and beyond C. In case *c*, the light is seen to be dispersed, that is, rendered divergent, after reflection. If these diverging rays enter the eye, the eye will form a *virtual* image behind the mirror, which will be erect and larger than the object for all positions of the object between F and P.

5. Image Formation by Refraction.—We shall next study the formation of images by *refraction.* The only refracted images that we shall consider are those formed by glass *lenses.* Lenses are of two general types, either convex, that is, thicker in the center than at the edges, or concave—thicker at the edges than in the center. (Fig. 80a and b.) When a plane-faced wave passes thru a convex lens, the center is retarded more than the edges so that the wave emerges as a *con-*

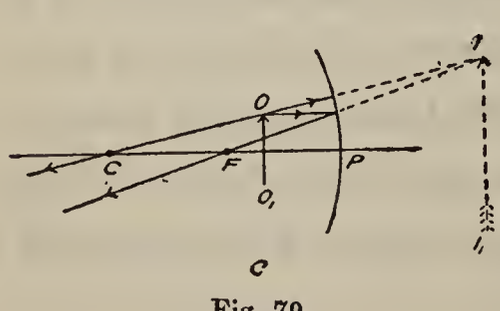

Fig. 79.

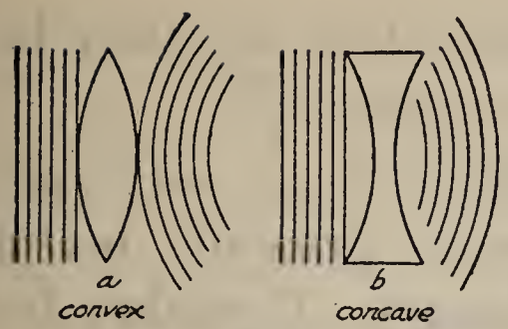

Fig. 80.

verging wave. Such lenses are therefore called *converging lenses* and correspond in an optical way to concave mirrors. When a plane wave passes thru a concave lens, the edges are retarded more than the center so that the wave comes out as a *diverging* wave and such lenses are called *diverging lenses* and correspond generally to convex mirrors.

It is evident that since concave lenses cause the incident light to diverge, they can never form real images. We shall therefore first study convex lenses, since they are the more important. We can see from Fig. 80a that after a plane-fronted wave has been rendered convergent by passing thru a convex lens, it must converge toward and pass thru some point on the axis of the lens. This point on the axis thru which a plane-faced wave with its face perpendicular to the axis passes after refraction, is called the *principal focus* of the lens. Speaking in terms of rays of light, the principal focus is seen to be the point on the axis of the lens thru which all rays which were parallel to the axis before refraction will pass after refraction. Let us call this distance from the principal focus to the center of the lens f. Now, in mathematics, the " curvature " of any arc is defined as the reciprocal of its radius—for example, the curvature of an arc drawn with a radius of 8 centimeters is $\frac{1}{8}$. It follows that the *curvature* of the converging wave coming out of the lens is $\frac{1}{f}$. We can therefore say that the *curvature which a lens impresses on an incident plane wave is* $\frac{1}{f}$. Now it is also a fact that no matter what may be the curvature of the incident wave, the lens will always

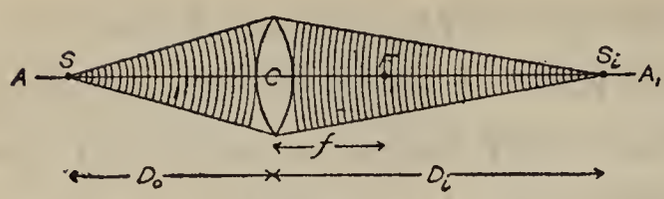

Fig. 81.

change that curvature by the amount $\frac{1}{f}$.

This is a very important principle, useful in calculating the position of images. Consider, for example, the waves issuing from the source S at a distance D_o from the center of the lens C (Fig. 81). Let these waves converge after passing thru the lens to the point S_i at distance D_i. S_i will be the image of S. Now the curvature which the wave has after passing thru the lens, that is $\frac{1}{D_i}$, is the difference between the curvature impressed by the lens, that is $\frac{1}{f}$ and that which the wave had when it entered the lens, $\frac{1}{D_o}$, or $\frac{1}{D_i} = \frac{1}{f} - \frac{1}{D_o}$. It is more usual to write this relation $\frac{1}{f} = \frac{1}{D_o} + \frac{1}{D_i}$. This formula is used to calculate the distance of an image formed by an object at a known distance from a lens of known focal length.

The focal length of a lens can always be found by holding it up in the sunlight. The image of the sun forms at the principal focus. It can be seen from the figure (Fig. 81) that if the source of light is put at S_i, the image must appear at S, as the conditions are exactly reversed. Such pairs of points, of which there are an indefinite number along the axis, are called *conjugate foci.*

6. Construction of Images: (a) Convex Lenses.—If we wish to find the position of the images by construction, we have two principles to guide us, just as we had with mirrors. The first principle has already been stated. It is that all rays parallel to the principal axis of the lens will after refraction pass thru the principal focus of the

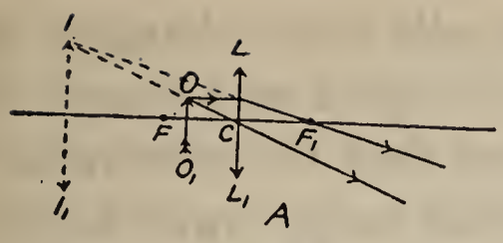

Fig. 82.

lens. The second principle is that any ray which passes thru the optical center c of a lens passes on undeviated, since whatever bending takes place at the first surface will be reserved at the second surface.

Figs. 82 and 83 show the application of these principles. For purposes of simplicity the lenses, LL_1, are represented by straight lines. In case A, the object is nearer the lens than the principal focal distance. The light is seen to be *divergent* after passing thru the lens. No real image can be formed. An eye would form a virtual image at II_1 which image is erect, larger than the object and on the same side of the lens with it. In case B, the object is a long way outside the focal distance. The image is seen to be real, inverted, smaller than the object and nearer the lens than the object. We can easily reason out the changes in size and position of this inverted real image with change in position of the object. If the object is at a very great distance, the image will be at the focus and will be very small. As the object is moved up toward the lens, the image will move out away from the focus and will become larger until finally both object and image are the same size. This will happen when both object and image are at a distance from the center of the lens equal to twice the focal length of the lens. If now the object is brought still closer to the lens, the image will move rapidly away and become larger until when the object is at the focus, the image is at infinity. If the object is taken inside the focus, we have case A again and the only image formed is a virtual one.

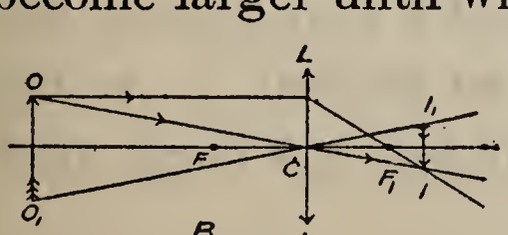

Fig. 83.

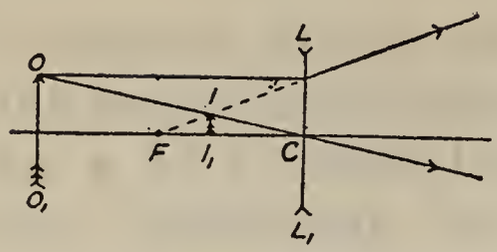

Fig. 84.

During all these changes in position of object and image, it will be seen that the extremities of object and image must lie between the lines O_1CI_1 and OCI so that object and image form corresponding sides of a pair of similar triangles. Since all corresponding dimensions of similar triangles are in the same ratio, it is seen that OO_1, the height or size of the object, must be to II_1, the size of the image, as the distance of the object D_o from the center of the lens is to the distance of the image D_i from the same point. This can be written $S_i : S_o :: D_i : D_o$ or $S_i\ D_o = D_i\ S_o$. This formula enables us to calculate the size of an image formed from an object of a given size, if we know or can calculate from the formula $\frac{1}{f} = \frac{1}{D_o} + \frac{1}{D_i}$ the distances of object and image.

(b) **Concave Lenses.**—We have already pointed out that concave lenses cause rays of light which pass thru them to diverge, so that no real images can be formed. In such lenses a ray parallel to the axis before refraction will after refraction proceed as if it came from that focus which is on the same side of the lens as the object—the focus being defined as the point on the axis from which an incident plane wave seems to proceed after refraction. Fig. 84 shows how these ideas are applied in constructing the virtual image given by a concave lens. This image is always erect and smaller than the object. It must be noted that the curvature of a wave passing thru a concave lens is *increased* by an amount $\frac{1}{f}$, so that the formula for the distance of object and image reads $\frac{1}{f} = \frac{1}{D_o} - \frac{1}{D_i}$.

7. Summary on Images Formed by Mirrors and Lenses.—We may sum up the information that we have gained in this section on images in the following form in which we group mirrors and lenses together:

(1) Virtual images are always erect. They are formed (a) by convex lenses and by concave mirrors, if the distance of the object from the lens or mirror is less than the focal length. Under these conditions the images are always larger than the objects. The resulting curvature of the wave front equals the initial curvature minus the curvature $\frac{1}{f}$ given by the lens or mirror, that is $\frac{1}{D_i} = \frac{1}{D_o} - \frac{1}{f}$. (b) By concave lenses and by convex mirrors under all conditions. These virtual images are always smaller than the objects. The curvature of the final wave equals the curvature of the incident wave *plus* the curvature $\frac{1}{f}$ given by the lens or mirror, that is $\frac{1}{D_i} = \frac{1}{D_o} + \frac{1}{f}$.

(2) Real images are always inverted. They are formed by convex lenses and concave mirrors if the distance of the object from the lens or mirror is greater than the focal length. If the distance of the object is greater than twice the focal length, the image is smaller than the object—otherwise it is always larger than the object. The curvature of the final wave equals the curvature given by the lens or mirror, $\frac{1}{f}$, minus the curvature of the incident wave—that is $\frac{1}{D_i} = \frac{1}{f} - \frac{1}{D_o}$.

(3) Size of images. The formula $S_i : S_o :: D_i : D_o$ gives the relation in size of object and image for all lenses and mirrors and all types of images.

8. Optical Instruments—the Camera.—We shall now describe a few optical instruments in which lenses are used. The first of these is the *photographic camera.*

This consists of a light-tight box in the front wall of which is a convex lens provided with a shutter that can be opened and closed. (Fig. 85.) At the back of the box is an arrangement for holding a photographic plate. This plate is at a distance from the lens slightly more than the focal length. When the shutter is opened, an inverted, diminished, real image of whatever is in front of the lens is formed on the plate. If the distances from lens to plate and lens to object are not in accordance with the relation $\frac{1}{f} = \frac{1}{D_o} + \frac{1}{D_i}$ the image will not be sharp—it will not be in focus. This can be corrected by moving the lens slightly. The light affects the emulsion on the photographic plate, so that after proper chemical treatment, the image is permanently fixed on the plate.

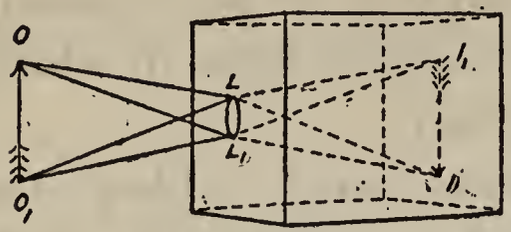

Fig. 85.

The human eye is very similar to the photographic camera. In front is a transparent lens, the crystalline lens, partly covered with an adjustable diaphragm, the iris. On the inner face of the back wall of the eye is the retina, a screen corresponding to the photographic plate. The fibers of the optic nerve spread out over this screen. The distance from the lens to the screen being fixed, such focusing of images as is necessary is brought about by an alteration in the curvature of the crystalline lens produced by a circular muscle around it. This change in the lens is referred to as the " accommodation " for objects at different distances.

The normal eye can bring parallel rays to a focus on the retina without strain. If the eyeball is too long, the rays will focus in front of the retina except when the object is very close to the eye. Such an eye is said to be

short-sighted or *myopic*. If an eyeglass is used in front of the eye consisting of a diverging (concave) lens of the proper focal length, the image may be made to fall on the retina. If the eye ball is too short from front to back, the images formed by the lens of the eye will fall back of the retina. The eye is then *far-sighted* or *hypermetropic*. This defect can be corrected by the use of a converging (convex) lens in front of the eye of such focal length that it will bring the image on the retina.

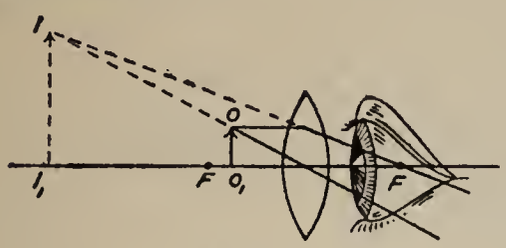

Fig. 86.

9. Magnifying Glass.—The magnifying glass in its simplest form is a single convex lens (Fig. 86). When in use one side is brought close to the eye and the object to be examined is placed at a little less than the focal distance on the other side. The image will then be virtual, erect and larger than the object. Since the image will appear most distinct when it is apparently 25 centimeters from the eye, the object has to be placed at such a distance from the lens that its virtual image is 25 centimeters from the eye. Since magnifying glasses are always of short focal length (2 to 5 centimeters) the distance of the object from the lens is roughly equal to the principal focal distance f. The distance of the image being 25 centimeters, we have $S_i : S_o :: 25 : f$ or $\frac{S_i}{S_o} = \frac{25}{f}$. This ratio is approximately the *magnifying power* of the lens. The formula furnishes a simple means of measuring magnifying powers when the focal length of a lens is known.

10. Telescope and Compound Microscope.—These instruments are essentially alike consisting of two converging lenses called respectively the *objective* lens L_o

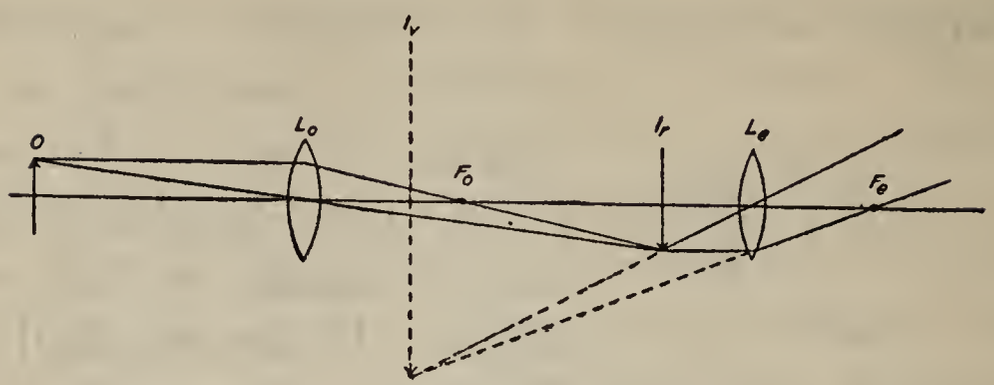

Fig. 87.

and the *eyepiece* lens L_e. In both instruments the objective receives the light from the object and forms a real image I_r which is received within the focal length of the eyepiece and magnified just as was the object in the magnifying glass of the last paragraph, forming a virtual image I_v. The figures (Figs. 87 and 88) show how the images are formed. Since the objects viewed with the telescope are always distant, the real image formed by the objective is very small and is practically at the principal focus. Objects viewed with the microscope are on the other hand always close up to the objective, close to the outer focal point of the lens. The image formed is therefore some distance beyond the inner focal point of the lens and is considerably larger than the object.

The telescope spoken of in the foregoing is the *astronomical telescope.* As can be seen from the diagrams, the final image is inverted with reference to the object. This inversion is not objectionable when studying stars, but when the telescope is used for terrestrial objects, the final image must be erect. In *terrestrial telescopes* the image is therefore made erect by introducing an extra converging lens between the objective and the eyepiece.

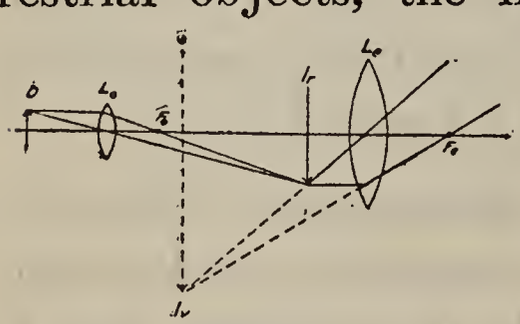

Fig. 88.

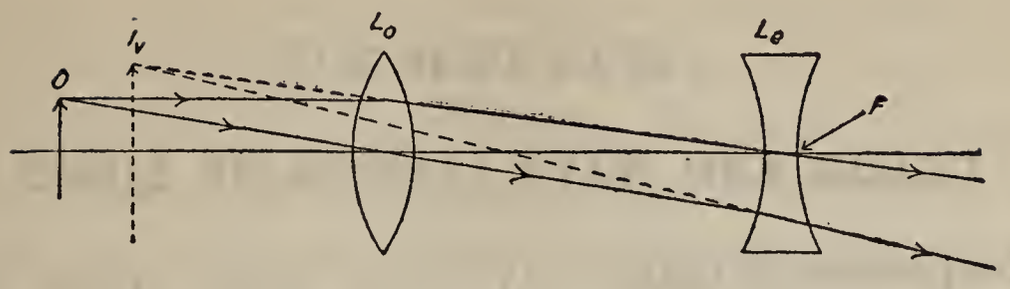

Fig. 89.

11. Opera Glass.—The opera glass or *Galileo's Telescope* has a converging objective like the microscope and telescope but before the converging rays form the real inverted diminished image, they pass thru a diverging (concave) lens and are rendered divergent. When these divergent rays enter the eye, the eye forms a virtual enlarged image which is erect with reference to the object (Fig. 89). This arrangement gives an erect image with only two lenses and with a comparatively short tube, so that in simplicity and size it is well adapted for general indoor use.

12. Prism Binocular.—The prism binocular generally used for military and field purposes is essentially an astronomical telescope in which the length of the tube is reduced by total internal reflection of the light between two pairs of prisms as shown in Fig. 90. The effective length of the tube is therefore three times its real length which gives us a useful combination of power with compactness.

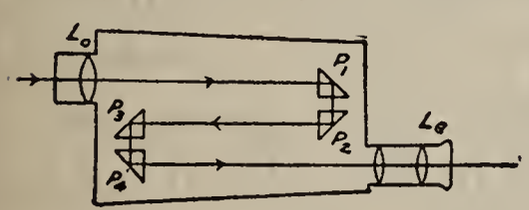

Fig. 90.

REVIEW.

1. Define real and virtual images.
2. What is the principal focus of a lens?
3. What are conjugate foci?
4. How is the magnifying power of a simple magnifying glass determined?

CHAPTER XII

COLOR AND WAVE LENGTH OF LIGHT

1. Dispersion of Light.—We now enter upon an entirely new phase of our subject—that of color effects. If a beam of sunlight admitted into a darkened room thru a hole in the window shutter is allowed to pass thru a triangular prism of glass, the beam will be refracted and bent around the base of the prism as our previous study would lead us to expect, but instead of a spot of light appearing on the wall after the bending, a band consisting of a whole series or " spectrum " of brilliant colors ranging from violet to red, like those in the rainbow, will be produced. This phenomenon is known as the " dispersion " of light. The blue light is bent the most and the red light the least of all the colors. This experiment shows first that sunlight is composite, containing light of many different colors, and second, that the amount of bending that a beam of light undergoes when refracted, depends on the color of the light.

Experiments similar to the one described in Chapter X where the wave-length of yellow light was determined from measurements of the bands or " fringes " produced by the interference of two reflected wave trains, have been used to find the wave lengths of the colored lights in the spectrum band given by a prism. The results for the wave length at the center of each specified colored region are:

Red	.000068 cm.	Yellow	.000058 cm.
Green	.000052 cm.	Blue	.000046 cm.
		Violet	.000042 cm.

We may now conclude that the refractibility or *refrangibility* of light depends on the wave length of the light: the shorter the wave length, the more *refrangible* the light—that is the farther the light will be bent when refracted by a given prism. The impression of color produced on the retina depends on the wave length of the light incident on it. Waves with a length of .000052 cm. will, for instance, produce a sensation of green in the normal eye. Color in light is thus seen to be analogous to pitch in sound. The sensation of green may however also be produced by a composite light beam containing a number of different colors blended—that is, containing a number of different wave lengths. We shall refer to this last matter again.

2. Colors of Opaque Objects.—The spectrum band produced by white or sunlight passed thru a prism is seen to be continuous, that is, to contain all wave lengths between roughly .00008 cm. and .00004 cm. This means that white light contains in it all the visible colors. If now an opaque body placed in sunlight appears red, it is evident that the light diffusely reflected from it must produce a sensation of red on the retina. Now when a beam of light falls on a body, it must be in certain proportions, absorbed, reflected and transmitted. In opaque bodies the amount *transmitted* is negligible, so that what is not reflected must be absorbed. It is evident that the opaque body which appears red in white light must have absorbed from the white light such constituents that the remainder, when reflected, produces a sensation of red on the retina. In general, then, the color displayed by an opaque body in white light depends on the selective absorption of its surface.

Suppose that the light diffused by this opaque red object is pure red—not a mixture producing the general sensation of red. Then the body must absorb all other colors, but red. If therefore such a body is illuminated with green, yellow, blue or light of any color other than red, it will absorb all the incident light and reflect nothing, and will appear black. The apparent color of an opaque object is consequently seen to depend in an important way on the color constitution of the light illuminating it. It is for this reason that ribbons and dresses frequently appear very different in color under artificial light from what they do in daylight. The mercury vapor or Cooper-Hewitt lamp so frequently used in photographic studios is entirely deficient in red light. The blood showing thru the skin of a person viewed under this flame, therefore, seems black or grey, producing a ghastly effect with which most of us are familiar.

It should be noted in this connection that a body which appears white in daylight, reflects all colors equally. Such a body viewed in a colored light will seem to be the color of the light—red in red light, green in green light and so on. A black body on the other hand is one which reflects no colors. It therefore looks black in light of any color as well as in white light.

3. Complementary Colors.—When an opaque body viewed in white light appears colored, it is reflecting to the eye certain wave lengths which produce the color sensation and it is absorbing certain other wave lengths which if they were emitted would produce a second different color sensation. These two sets of waves striking the retina at the same time would of course be, and produce the effect of, white light. Such pairs of colors

—such pairs of groups of wave lengths, which if received simultaneously on the retina would produce the effect of white light, are called " complementary " colors. Whatever wave lengths are subtracted from white light to produce one color sensation, the remaining wave lengths will produce the sensation of the complementary color.

It is evident that there is an indefinite number of pairs of complementary colors. Among these are yellow and blue—green and crimson—red and bluish green—orange and greenish blue—violet and greenish yellow.

The complement of any color, as, for example, green, can be found experimentally as follows. Stare fixedly for 20 or 30 seconds at a green spot, $\frac{1}{2}$ inch in diameter, in the center of a white card. Then transfer the fixed gaze to a sheet of white paper. A crimson spot will appear in the center of the white sheet. The explanation of this fact is that the area of the retina on which the green image fell became fatigued by the green sensation during the time of staring at the green spot. When waves of all lengths from the white paper fall on the retina, there is no response or only a weaker response to the waves producing the green sensation, so that the effect registered is that of white light minus the green—in other words the complement of green, which is crimson.

4. Mixed Colors and Mixed Pigments.—If one looks at a rapidly rotating paper disk, colored in two sectors with any pair of complementary colors, the disk will appear white or grayish white because the two color impressions will follow one another so rapidly on the retina that they will blend into a single sensation. Any combination of colors desired may be arranged on the disk and on being mixed by rapid rotation, a sensation of some resultant color will be produced.

The rotating disk actually "mixes colors" in that it superposes on the retina the effects of the different groups of waves belonging to the individual colors on the disc and thus produces a composite effect. This is quite a different thing from mixing pigments or paints on a paper. If yellow light is added to blue light of the proper shade, white light is produced, but if yellow paint is mixed with blue paint, green paint results. This is because the yellow paint removes the blue and violet from white light by absorption and the blue paint removes the red and yellow so that only green is left in the light reflected. The color of a mixture of pigments depends, then, on the light which escapes absorption by the constituents of the mixture.

The apparent color of transparent objects depends on the light which they transmit—the remainder of the incident light being absorbed or reflected. Red glass, for instance, looks red by transmitted light because it absorbs all the other constituents of white light. It looks red by reflected light because of the fraction of the light which penetrates a small distance into the glass and then comes out again. If the light incident on the glass contains no red waves, the glass appears black both by transmitted and by reflected light.

5. The Spectroscope.—In order to find out the wave lengths contained in the light given by different sources, an instrument known as a spectroscope is employed. This instrument consists of three parts, a collimator, a prism and a telescope, all mounted on one base. The collimator is a tube which has at one end a narrow slit and at the other end a convex lens. The slit is at the focus of the lens so that when the slit is illuminated with the light to be examined, a parallel beam is delivered

from the lens. This beam is directed on the prism which refracts it and disperses it into a spectrum. This spectrum enters the telescope which is optically an ordinary astronomical telescope where a real image of the slit for each color coming thru the slit is formed, and these slit images are magnified by the eyepiece.

If the light incident on the slit is white, the slit images lap each other side by side, so that a continuous graduated band of color is produced. Such a spectrum is called a *continuous spectrum.* Spectra of this sort are produced by white hot solids and liquids. If the incident light lacks certain wave lengths the spectrum will be broken up into colored bands or lines and is then called a *bright line spectrum.* Incandescent gases and vapors produce such spectra which are very characteristic of the materials contained in the gas or vapor—so characteristic, in fact, that one of the most delicate methods of chemical analysis is based on a study of the bright line spectra produced by the substance to be analyzed when volatilized in a flame.

Lastly, if white light is passed thru a material which absorbs some wave lengths from it, before it enters the slit, then a continuous spectrum is produced with black lines across it, wherever absorption has taken place. Such a spectrum is called a *dark line* or *absorption spectrum.*

An examination of the spectrum produced by sunlight shows it to belong to the class of absorption spectra. It is crossed by many hundred fine black lines called, after the man who first carefully studied them, the *Fraunhofer lines.* Most of these lines are formed by the absorption of certain wave lengths from the white light from the sun's incandescent surface as this

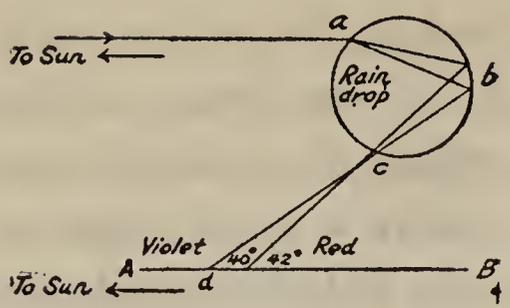

Fig. 91.

light passes out thru the somewhat cooler outer atmosphere of the sun. It is from study of these lines that the materials which make up the atmosphere of the sun have been determined. Some of the Fraunhofer lines are produced by absorption in the earth's atmosphere.

6. The Rainbow.—The rainbow is a spectrum of sunlight frequently seen in Nature. The rainbow can be seen only when the sun is not more than about 40° above the horizon, when the observer has his back to the sun and drops of rain are falling before him. If (Fig. 91) a light ray enters a rain drop at *a* it is refracted to *b* where it is reflected to *c* and passes out along the line *cd*. On its path thru the rain drop, the ray is dispersed and a spectrum formed. The angle of the red ray in this spectrum with a line *AB*, parallel to the incident rays from the sun, will be 42°; the angle of the violet rays 40°.

If now a man stands at *P* on the ground (Fig. 92) looking up at a screen of rain drops, the drops at an angle of 42° in an arc around a line drawn thru his eye toward the sun will appear red—those at an angle of 40° will appear violet and intermediate drops will appear of intermediate colors—thus forming a rainbow. This kind of bow is called a *primary* rainbow. A spectrum can also be formed from a spherical drop by two internal reflections, as shown in the upper two drops in Fig. 92. The red rays in this case are thrown out at an angle of 51° and the violet rays at an

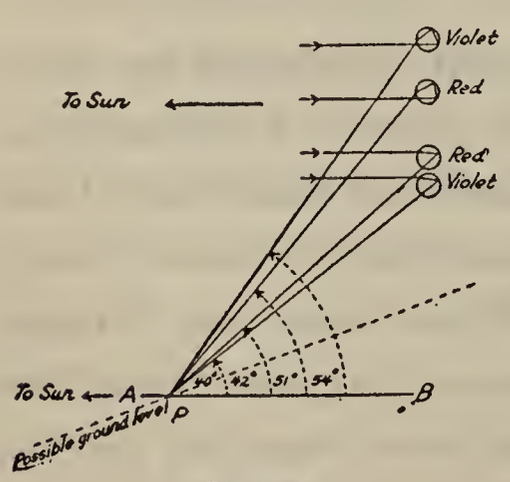

Fig. 92.

Fig. 93.

angle of 54°, with the line *AB*. The resulting rainbow, called a *secondary* bow, will therefore appear above or outside the primary bow and will have its colors reversed in order as shown. On account of the double internal reflection, this bow will be fainter than the primary bow and is not so often seen.

7. Chromatic Aberration.—Before leaving this section of our subject, there is one other matter of practical importance, which must be referred to; that of the so-called " chromatic aberration " of lenses. Since a convex lens is, in a way, similar to a pair of prisms set base to base, it is evident that light passed thru such a lens must be dispersed so that the images formed will be diffused and colored. Since the blue light from the source is bent more than the red light, it is evident that the blue light will focus nearer the lens than will the red. This is the phenomenon known as chromatic aberration.

It would be impossible to correct this difficulty if the dispersive powers and the refractive powers of all materials were related in the same proportion. It is found, however, that some glasses have much higher dispersive powers than others of nearly the same refractive power. For example, crown glass has a dispersive power of .21 and a refractive index of 1.5 while flint glass with a refractive index of 1.6 has a dispersive power of .45—more than twice that of crown glass. If therefore we combine a convex lens of crown glass with a thinner concave lens of flint glass, as shown in Fig. 93, we can produce in

the first lens, both bending and dispersion and in the second lens almost completely overcome the dispersion without completely overcoming the bending. Such a combination is called an *achromatic* lens. The images formed by achromatic lenses are sharp and without colors due to aberration. Such lenses are used in all good optical instruments.

8. Long and Short Waves—the Radiation Spectrum.—There are limits to the sensitivity of the eye to color, just as there are limits to the sensitivity of the ear to pitch. Ether waves shorter than those of the extreme violet (about .00003 cm. long) and longer than those of the extreme red (about .00008 cm. long) have no effect on the retina. There is however a long series of such ether waves which can be detected by other effects which they produce. All ether waves, including those of light, are grouped together under the head of " radiation " and the series of these radiations arranged according to wave length is called the " radiation spectrum." The rate of propagation of all these waves is the same—namely 186,000 miles per second. Longer than the red of the solar spectrum, we have a series of invisible radiations, ranging to wave lengths of .03 centimeter and longer—more than 400 times as long as the longest red waves. This is called the *infra-red* portion of the spectrum. These waves, when absorbed in matter, are very effective in producing heat motion of the molecules and are therefore frequently called " heat rays." However, all lengths of radiation will produce heat effects when absorbed, so this name is somewhat misleading.

Still longer than infra-red rays, ranging from .3 centimeters to many kilometers in length, are the *elec-*

trical or *Hertzian* waves. These waves are produced by electrical discharges and are detected by their electrical effects. Their position in the radiation spectrum was first established by Heinrich Hertz, of Germany, in 1888. These are the waves used in transmitting wireless telegraph and wireless telephone messages.

On the other end of the solar spectrum, shorter than the violet, we have the *ultra violet* region in which the radiations have wave lengths which have been measured down to .000003 cm.—about one-eighth of the shortest violet rays. The most important work in this portion of the spectrum has been done by Professor Lyman, of Harvard University.

Still shorter than the ultra violet radiation are the waves given out by certain materials when struck by discharges of " electrons "—which are rapidly moving negative charges of electricity, about which we shall hear more in the next chapter. These waves include the different series of X-rays. They have, in very recent years, been measured down to .000000006 cm. The shorter of these rays pass readily thru most ordinary bodies—such as the human flesh, wood, fabric, etc. All short wave lengths, including the ultra violet as well as the X-rays, are very active chemically and are readily detected thru their influence on the photographic plate. Excellent pictures can be taken with them by specially prepared cameras. These rays are consequently frequently called *chemical* or *photographic rays.*

It will be noted that visible light constitutes only a very small portion of the entire radiation spectrum. This spectrum comprising as it does, the X-rays, the chemical and photographic rays, the so-called heat rays, and the electrical rays of wireless telegraphy, has numer-

ous applications of the greatest importance in modern life.

REVIEW.

1. On what does the color displayed by an opaque body in white light depend?
2. What are complementary colors?
3. Explain chromatic aberration.
4. What are Hertzian waves?

CHAPTER XIII

STATIC ELECTRICITY

1. Electrification.—Electricity has done more than any other one agency to bring about those great changes in conditions of living which make our age differ from preceding ages.

Electricity may exist in a stationary state, collected on a body, or in a continuously moving state, passing along a wire or similar conductor. It is in the first case known as static electricity, and in the second as current electricity. These two types are identical—static electricity in motion constituting current electricity. Electricity in motion has many more important applications than has electricity at rest. We shall consider briefly the phenomena of static electricity before passing on to the subject of current electricity and the associated facts of magnetism.

If a hard rubber fountain pen be rubbed briskly on the sleeve of a woolen coat it will acquire the power of attracting and picking up small pieces of paper and other light objects. A stick of sealing wax rubbed with flannel or a piece of glass rubbed with silk will acquire the same power. A body possessing this power is said to be *electrified.* Electrification by rubbing was first observed by Thales, a Greek philosopher (600 B. C.), in dealing with amber. That the same effects are produced by rubbing many different substances was discovered by Dr. William Gilbert (1540-1603), physician to Queen Elizabeth of England.

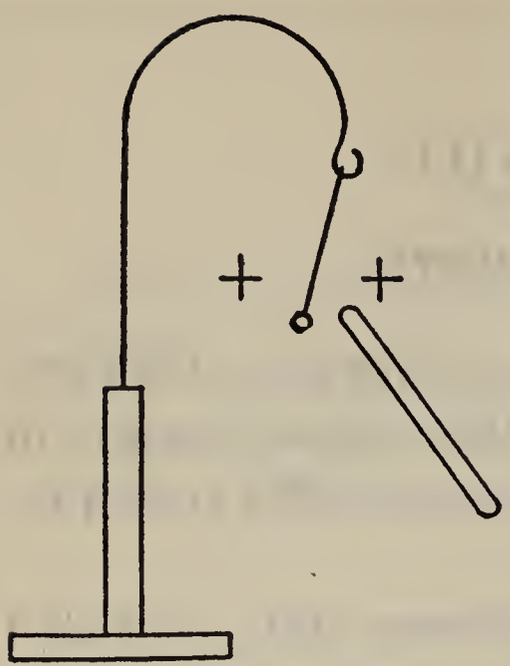

Fig. 94.

2. Positive and Negative Electricity.—Several interesting facts about electrified bodies can be brought out by the use of a pith-ball suspended from a silk thread as shown in Fig. 94. Pith is selected because it is very light in weight. If a bar of sealing wax is rubbed with flannel and brought near the pith-ball, it will at first attract the ball until the two touch—then a definite repulsion will be observed. If now a glass rod rubbed with silk be brought up to the ball, the ball repelled by the wax will be attracted by the glass. The character of the electrification on the glass and on the wax is thus seen to be different. All electrified bodies brought near the pith-ball can be classified into two groups, depending on whether they attract it like the glass or repel it like the wax. To distinguish these two types of electrification one, that on the glass, is called *positive electrification* and the other, that on the wax, is called *negative electrification.* An electrified body is said to be *charged* with *positive electricity* or with *negative electricity* according to the character of the effects —that is, whether it attracts or repels the negatively charged pith-ball.

If two electrified bars of wax are suspended from silk threads and brought near together, they will repel one another; but if a suspended electrified glass rod is brought near one of the bars of wax a definite attraction will take place. It will, in fact, be found in all cases that bodies *electrified with the same kind of electricity repel*

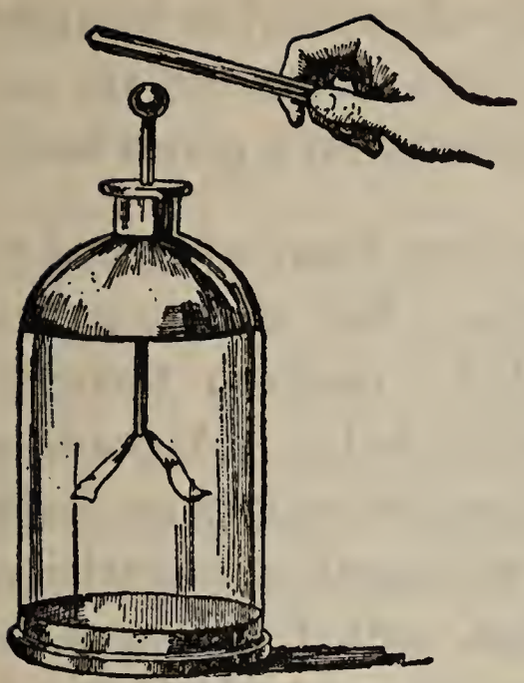

Fig. 95.

each other and that *bodies electrified with opposite or dissimilar kinds attract one another.* Further, the strength of these attractions and repulsions depends on the quantity of electricity on the bodies, and it *varies inversely with the square of the distance* between the bodies, just as does the intensity of sound and light.

The forces change in magnitude if the medium between the charged bodies is changed—if, for instance, sealing wax be substituted for air. When the force actions are increased in any medium above their value in air for the same charges and distance, the medium is said to have a lower "specific inductive capacity" than air. The specific inductive capacity of air is taken to be *one.* In this Text we shall consider the medium always to be air.

These laws of attraction and repulsion are made use of in the *gold leaf electroscope,* a device for detecting the presence of electrical charges and for indicating their kind. It consists of a wire with a ball or disc at the upper end and a hook supporting a doubled strip of gold leaf at the lower end (Fig. 95). The wire is held in the cork of a bottle as shown in the figure. If the wire is charged by touching it with a rubbed bar of wax, the two pieces of gold leaf will repel one another and diverge, thus indicating the presence of a charge. The leaves will in this case be electrified with negative electricity. If now, a rod, positively electrified, be brought near the disc without touching it, the leaves will at first

fall together; but if one negatively electrified be brought near, the leaves will diverge farther. Thus we are enabled to identify the kind of electricity on a given body.

3. Electrons.—Various theories have been advanced to explain the facts of electrification. We shall concern ourselves only with the so-called "electron theory," which is the one generally accepted today. According to this theory, the atoms of all kinds of matter are built up of charges of positive and of negative electricity. The central portion of the atom, called the *nucleus,* which constitutes most of its mass, is positively charged. Surrounding it are numerous small negatively charged *electrons,* each of which has roughly $\frac{1}{1800}$ the mass of a hydrogen atom. In an uncharged atom the total negative charge on the electrons exactly equals the positive charge on the nucleus.

The electrons of all forms of matter are identical in character. They usually are supposed to be in motion with respect to the nucleus, revolving around it, but the nuclei themselves are supposed to remain fixed in position in the body of which they are constituents. In some kinds of matter, called insulators or dielectrics, the electrons seem firmly bound to the positive centers; in other kinds of matter, called conductors, they move about more or less freely from atom to atom. Glass, rubber and sealing wax are good insulators; brass and other metals good conductors. When a body is uncharged with electricity the sum of all the negative charges of the electrons exactly equals the sum of all the positive charges on the atomic centers, so that there are no resultant effects. If a body has a larger number of electrons on it than is necessary to equal the positive charges, it will exhibit negative electrification; if it has

a less number it will exhibit positive electrification. If a negatively charged body is connected with an uncharged or with a positively charged body by a wire of a material thru which electrons can pass, that is by a conducting wire, the excess electrons will pass along the wire to the positively charged body until the number of electrons in unit volume is the same on both bodies. This flow of electrons constitutes an *electric current.* Electrons, being negative charges, repel one another but are attracted to the positive centers.

4. Frictional Electricity.—All the observed facts of electrical science can be explained, generally with complete success, on the basis of this theory. Electrification by friction is explained in the following way: When two bodies are rubbed together there will be a redistribution of electrons and some will be moved from the one body to the other. This will leave a positive charge on the one body and an equal negative charge on the other. It can be shown experimentally that when glass is rubbed with silk, the silk acquires a negative charge equal to the positive charge on the glass, and so generally the rubber gains always a charge opposite to that of the object rubbed. As the rubber usually is held in the hand, the charge on it leaks away to the ground, over the hand and body, and so cannot be detected unless special precautions are taken. Glass being a good insulator does not afford a free path for electrons, so that a charge produced on one end of a glass bar will remain there, even when the other end is held in the hand. So also with sealing wax and other insulators. If, however, a metal bar be held in the hand and rubbed, the electrons flow freely along it from or to the hand and thus no final electrical effect can be developed. If the metallic bar

be supported on insulators, then a frictional charge can be produced on it. This charge will instantly distribute itself evenly over the conducting metal and will not stay on the rubbed area as it does when insulators are rubbed.

5. Distribution of Electrons.—In studying electrical effects, it is important to remember that the positively charged atomic centers remain fixed, while the negative electrons move. The distribution of charge on a body therefore depends on the distribution of the electrons. Since these repel one another, it is evident that when the charged body is a conductor, that is, a body on which the electrons move about freely, the excess electrons will be found as far apart as they can get, that is, they will be on the surface of the body. The free electricity on a charged conductor is then on the surface of the body and not in the interior. Therefore, when an electrical charge is communicated to a conductor, it distributes itself over the surface. The density of the distribution is, however, not uniform unless the curvature of the surface is at all points the same. Where the curvature is greater, there the density is also greater. The density of charge will consequently be greatest at sharp points on the surface. If the density becomes great enough, the charge will leak off into the air which, ordinarily an excellent insulator, is made a conductor thru the extraction of electrons from its atoms by the attraction or repulsion of the charge on the point.

When the air is in this conducting state, containing free electrons, it is said to be " ionized." On account of this action, it is difficult to keep a charge on a body which has sharp points or angles. The protective action of lightning rods depends on this fact. When two bodies charged, one positively and the other negatively, are

brought near together, the densities of charge will increase very rapidly on the parts of the bodies which are closest together, until the air is ionized, when the electrons will pass across from the negative to the positive body. This passage will be accompanied by a spark and a cracking sound. During thunder storms heavy electrical charges are developed in the clouds, which affect the distribution of electrons on the surface of the earth beneath in such a way as to develop great densities of charge, especially on pointed objects such as church towers or trees. If this density becomes great enough, a discharge—a lightning stroke—takes place with an associated crash of thunder. If the church tower be provided with a number of sharp points connected with the ground by conducting rods these may sufficiently reduce the density of charge by continuously discharging it into the air and so eliminate the chance of lightning striking. This device was introduced by Benjamin Franklin (1706-1790), who was first (1752) to show that lightning is essentially an electric spark.

It must be noted that lightning rods are not intended to conduct the lightning to the ground but merely to discharge the electricity from the areas to which they are connected. When properly installed they serve a useful purpose; but as they not infrequently have failed to protect the buildings to which they have been attached from discharges of certain kinds and intensities, they have, in recent years, fallen somewhat into disrepute.

6. Potential.—If when two charged bodies are connected by a conducting wire a discharge of electrons passes from one body to the other, a "difference of electrical potential" is said to exist between the two

bodies. It is the custom to refer to the potential of the body with the excess of electrons as a negative potential with reference to the other body. Its potential is said to be lower than that of the positively charged body. Since it seems natural to think of the discharge as passing from the high potential to the low potential, it is customary to say that discharges do so pass, but it must be remembered that according to the electron theory the flow of electrons must always be in fact from the negative to the positive, that is, from the "low" to the "high" potential.

A body is said to be at zero potential when it is at the same potential as the earth, so that when connected to the earth no electron flow takes place in either direction. If it has more electrons on it than meet the above condition, it is said to have a negative potential—to be at a lower potential than the earth; if it has less electrons than necessary to give it zero potential, it is said to be at a positive potential—higher than the earth.

7. The Voltaic Cell.—If we can maintain two plates or other conductors steadily at different potentials, after having connected them with a conducting wire, we can keep up a steady flow of electrons from the negative plate to the positive plate. This flow will constitute a *current of electricity.* There are several ways in which this steady difference of potential can be maintained. The first to be discovered was the chemical method.

If two conducting plates are placed in a liquid which attacks one of them chemically but does not affect the other, the plate which is attacked will have the lower potential. This was first observed by Galvani, an Italian anatomist, in 1786, but Volta, another Italian, was the first to apply the idea successfully to a current gen-

erator. This he did in 1800, producing the device now known as the Voltaic or galvanic cell.

The cells can be made up of a great variety of materials. A typical combination consists of a strip of copper and a strip of zinc immersed in dilute sulphuric or hydrochloric acid. The acid dissolves the zinc but not the copper. The zinc is always at a lower potential than the copper. If the plates are connected outside the liquid with a metallic wire, a current flows which is said to pass from the copper to the zinc, altho as we have already pointed out, the electrons actually flow from the zinc to the copper.

The current flowing in the wire develops certain effects which are not developed by stationary electric charges. Chief among these is the magnetic effect. Oersted (1777-1851), a Danish physicist, discovered, in 1820, that if a wire bearing a current is placed above and along the line of a compass needle, the needle will be strongly deflected toward a position at right angles to the wire; and in 1828 Joseph Henry (1797-1878), an American, found that a current-bearing wire wound round an ordinary iron bar converts the bar into a powerful magnet. Currents of electricity can therefore be easily detected by the magnetic effects which they produce. Before continuing our study of electric currents we must take up, briefly, the properties of magnets.

REVIEW.

1. State the electron theory.
2. How does a lightning rod protect a building against being struck by lightning?
3. What is meant by a difference of electrical potential between two bodies?

CHAPTER XIV

MAGNETISM—OHM'S LAW

1. Magnets.— Magnets are either natural or artificial. Natural magnets are fragments of a certain iron ore known as magnetite. Artificial magnets are pieces of iron or steel which have been rendered magnetic either by stroking with a natural magnet or by passing an electric current thru a coil wound about them as described in a preceding paragraph.

Magnets are distinguished by possessing the power to attract and pick up pieces of iron or steel, and by their tendency to swing into a north and south direction when freely suspended. Natural magnets have been used for many centuries as compasses or lodestones (leading stones). The compass was first used in Europe about 1190, but was apparently known in China before that date. The end which points north is called the north-seeking pole or the north pole; the end which points south is called the south-seeking or south pole. If two suspended magnets are brought near together, the two north poles or the two south poles repel each other and the unlike poles attract each other. The magnitude of these attractions and repulsions will vary inversely with the square of the distance between the poles.

A pole which repels an equal opposite pole placed at a distance of 1 centimeter in air, with a force of one dyne, is said to have a strength of one unit and is called a "unit pole." If the surrounding medium is not air,

the force effects will in general be different from those in air, even if all other conditions remain the same. This fact is covered by saying that different media have different *permeabilities* to magnetic forces. In our work all the poles talked about are assumed to be in air.

If a piece of soft iron be held near the pole of a magnet, it becomes a magnet by induction—the end of the soft iron nearest the inducing pole becoming a pole of the opposite kind. The resulting attraction draws the two together. If the soft iron be removed from the neighborhood of the inducing magnet, it at once loses nearly all its magnetism. We say that its *retentivity* for magnetism is low. Steel, on the other hand, altho harder to magnetize, has a high retentivity and if once made magnetic remains so.

If a magnet be heated red hot or be hammered or twisted, it loses its magnetism. If a long bar magnet be broken into short pieces, each piece will be a magnet with a north and a south pole. These, and other known facts, indicate that magnetism has to do with the arrangement of the molecules inside the magnet. The theory of magnetism generally accepted today is that the molecules of magnetic substances are themselves small magnets. In an unmagnetic bar these tiny magnets are arranged at haphazard so that opposite poles neutralize each other thruout the bar. If such a bar be brought near a magnet, the external magnetic forces swing the little magnets around into line with all the north poles in one direction, so that the bar then possesses polarity, one end being a north pole, the other end a south pole. When all the little magnets are in line, the maximum strength of magnetization will be

reached. Such a point can be reached with all bars, and, magnetized thus to their limit, they are said to be *saturated.*

2. Magnetic Field of Force.— The space around a magnet thru which its force effects can be perceived is called the *field of force* of the magnet. The magnitude and direction of the force at any point in this field can be investigated by imagining that we have at that point a "free north pole," that is to say, an isolated north pole without any associated south pole. From the very nature of magnetism, as already explained, it is impossible actually to have such a "free pole," since for every north pole there must be a corresponding south pole; but it will help us if we imagine that we have one.

If the free north pole is released at any point, it will move along some curve under the repulsion of the north pole of the magnet and the attraction of the south pole until finally it reaches the south pole of the magnet. The path which it traces is known as a *line of force.*

It is evident that there is an indefinite number of lines of force in any field of force. The direction of the "line" at any point indicates the direction of the resultant magnetic force at that point. If a small compass needle, that is, a short bar magnet pivoted on a sharp point, be placed at a point in a field, it will turn until its length lies along the line of force at that point. The needle will not move endwise since the pulls on its two poles are equal and opposite. If iron filings are scattered on a sheet of glass under which a magnet lies, each small filing will become a magnet by induction and will turn along the lines of force at the point where it

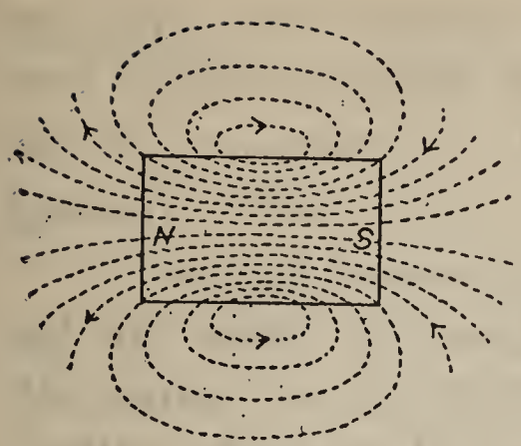

Fig. 96.

fell, just as a compass needle would. A regular figure, like Fig. 96, will result, showing the lines of force in all parts of the field.

3. Terrestrial Magnetism.—That a suspended magnet turns always into a north and south line indicates that the earth is surrounded by a magnetic field. As a matter of fact, the magnetic effects around the earth are the same, speaking generally, as if a long bar magnet were thrust thru the earth with one pole coming to the surface in Boothia Felix, Canada, latitude 70° 5′ N., longitude 98° 46′ W. A freely suspended needle carried to that place, called the *north magnetic pole* of the earth, will point vertically up and down. This pole was discovered in 1831 by Sir James Ross.

The geographic north pole—the end of the earth's axis—is of course at latitude 90°. A compass needle between the north geographic and the north magnetic poles will point to the north magnetic pole, that is, to the geographic south. So, generally, a compass needle points, not to the true or geographic north pole, but to the magnetic north pole. In the near neighborhood of the magnetic pole it may point north, south, east or west, depending on its position relative to the magnetic pole. In parts of the earth distant from the pole, the deviation of the compass needle from the true north is not so marked; but, except on a line drawn thru the geographic and magnetic poles, there is always some deviation to the east or west. This deviation is known as the *declination* of the compass. It is of the highest importance for mariners to know this declination, which

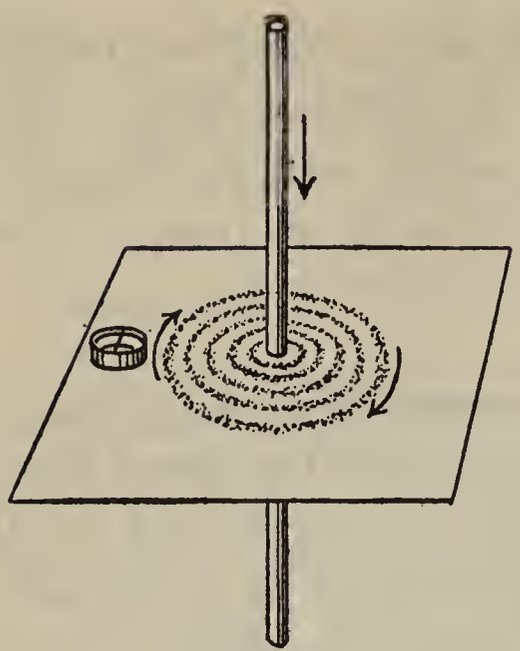

Fig. 97.

is not constant at any one place but changes slowly over a period of years, as if the magnetic pole were moving slowly in a closed curve.

The lines of force in the earth's field are of course directed towards the poles. They will be, in general, parallel to the earth's surface only at the magnetic equator. Elsewhere they strike down into the earth. A freely suspended compass needle will always lie along the line of force at the point and will therefore always point downwards toward the earth excepting on the magnetic equator. This inclination is known as the *dip* of the compass. At Washington this dip is about 70°. As already stated it is 90° at the magnetic poles. The ordinary compass needle is so supported on its pivot that the dip is corrected for; the directing influence coming from the horizontal component of the earth's magnetic force.

4. Oersted's Discovery.—We are now in a position better to understand the significance of Oersted's discovery, that a compass needle placed near a wire bearing a current of electricity is deflected toward a position at right angles to the wire. It is evident that the wire must be surrounded by a magnetic field of force, the lines of which lie more or less at right angles to the wire. They are, in fact, in the form of concentric circles about the wire (Fig. 97). Their direction, that is, the direction in which a free north pole would move around the wire, if released near it, can be

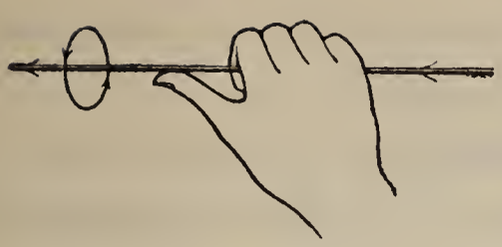

Fig. 98.

told by the following Right Hand Rule: Take the conductor in the right hand, as shown in Fig. 98, with the thumb pointing in the direction of the current flow—that is from the positive to the negative plate of the driving cell—then the fingers of the hand bent around the wire will lie in the direction of the lines in the magnetic field of the current.

5. Galvanometers.—The intensity of the magnetic force at any point near a wire, which is defined as the number of dynes of force acting on a free unit magnetic pole at that point, is directly proportional to the current strength, that is, to the amount of electricity passing thru any cross section of the wire in one second. The current can therefore be measured by measuring the intensity of the magnetic field produced by it. A device designed to make such measurements is called a *galvanometer*.

Galvanometers are of two general types. The first, the suspended needle type, has a stationary coil of relatively large diameter, generally 5 to 12 inches, thru which the current to be measured is passed. At the center of the coil a magnetic needle is suspended which is deflected from its normal north and south position by the field produced by the current. The second, the suspended coil type, has a small rectangular coil thru which the current is passed. This is suspended by a fairly stiff wire in the strong field between the poles of a permanent magnet. The magnetic field set up by the current passing thru the coil reacts with the field of the permanent magnet, and the coil turns, being subsequently brought back to its original posi-

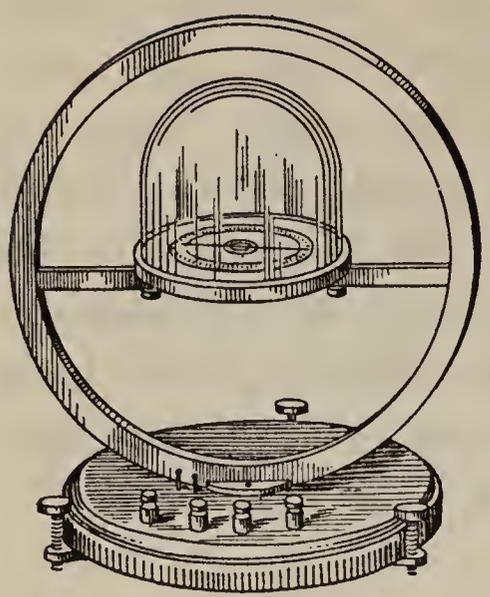

Fig. 99.

tion by the elasticity of the suspending wire.

The *tangent galvanometer,* Fig. 99, is of the suspended needle type. The large stationary coil is set always with its plane north and south, so that the lines of the field produced by the current flowing in it will be east and west at the center of the coil. At the center there will be a resultant magnetic field built up of two components—the earth's field north and south, and the field due to the current east and west. The small compass needle at the center of the coil will turn always so as to lie along the lines of the resultant field. The amount of the deflection is taken as the measure of the current strength.

The unit of current is called the *ampere.* An ampere flowing in a tangent galvanometer coil of three turns of 10 centimeters radius will deflect the needle 45 degrees at a place where the horizontal component of the earth's field is the same as at Washington, D. C. The ampere is accurately defined as the current which flowing in a coil of one turn with a radius of 1 centimeter, will produce a magnetic force action of $.2\pi$ dynes on a unit pole placed at the center of the coil. A galvanometer provided with a scale which reads directly in amperes is called an ammeter. Ammeters are usually of the suspended coil type, as these instruments are more robust and less subject to disturbance by outside fields than are suspended needle instruments.

6. Electrometers.—The difference of potential developed between the two plates of a galvanic cell, before

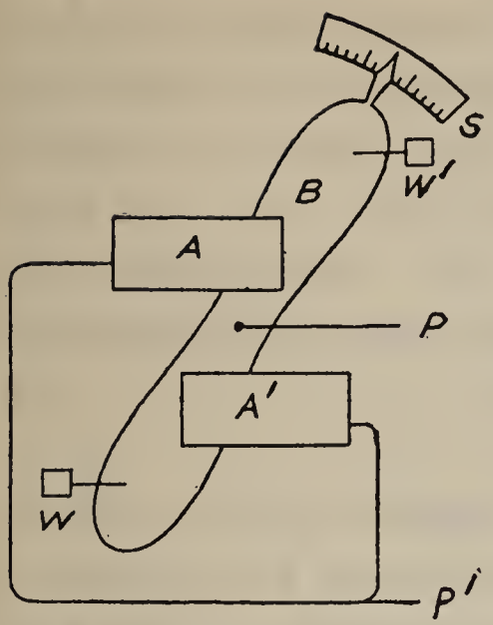

Fig. 100.

they are connected with a wire, depends, speaking very roughly, on the relative rates of solution of the two plates in the liquid. Cells made up of different materials develop consequently various differences of potential, which are dependent in any case only on the materials used and not at all on the size of the plates or on the distance between them. We can compare the potential differences of different cells by connecting the plates to an instrument known as an electrometer. One type of this instrument is shown diagrammatically in Fig. 100. The paddle-shaped metal piece B is arranged so as to turn freely on an axis thru its center. It is held in a definite position by weights hung at W and W^1. A and A^1 are conductors bent so as to inclose B *partly*. If P and P^1 are connected to the plates of a galvanic cell, the paddle will, on account of the electro-static attractions, be drawn further into the curved pieces A and A^1 by an amount depending on the potential applied. This movement is read on the scale S.

The unit of potential difference is called the *volt*. It is defined as $\frac{1}{1.0183}$ of the potential difference developed between the plates of a special kind of galvanic cell known as the Weston normal cell. The positive plate of this cell is of mercury in a paste of mercurous sulphate—the negative plate of cadmium amalgam in a saturated solution of cadmium sulphate (Fig. 101). As indicated in our definition for the volt, this cell develops a potential difference of 1.0183

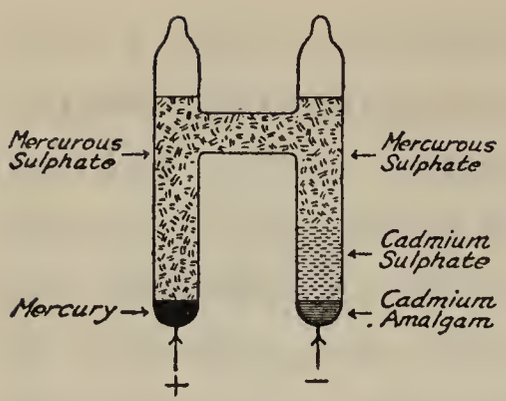

Fig. 101.

volts. This particular kind of cell is selected as a standard because it is easy to reproduce accurately and is very constant in developing always the same difference of potential.

7. Electro-Motive Force.—The capacity of a cell or other generator of currents of electricity to produce a difference of potential between its terminals, is called its "electro-motive force" or, for short, e. m. f. According to the electron theory the two plates are at different potentials because one of them has more electrons per unit of volume than the other. If we connect the two plates, electrons will begin to flow from one plate to the other, and unless the electrons withdrawn from the negative plate are replaced from inside the cell, as fast as they are drawn off outside, the difference of potential between the plates will evidently become less.

It is clear then, that the potential difference between the plates of a cell is not necessarily the same when a current is being drawn from the cell, as it is when the plates are not connected with each other—or, in electrical phraseology, are "on open circuit." The potential difference developed by a cell on open circuit measures the e. m. f. of the cell; and this depends, as we have already said, only on the materials of the cell. When the cell is on closed circuit—that is, when its plates are connected with each other thru any conductors—the potential difference between the plates depends on the relative ratio of withdrawal and of replacement of electrons on the negative plate. If the rate of withdrawal be small compared with the rate of replacement,

as is usually the case in actual practice, the change in potential difference at the plates, due to the current flow, is small or even negligible.

8. Voltmeters.—If a galvanometer be made with long coils of fine wire so that a negligible amount of current flows thru it when it is connected to the plates of a galvanic cell, its deflections may be used to compare the electro-motive forces of the cells, since whatever small current does get thru must be proportional to the potentials applied to the ends of the coil. An instrument so constructed and provided with a scale reading in volts, is called a *voltmeter*. In industry potential differences and e. m. f.'s are nearly always measured with such instruments.

9. Conductivity and Resistivity.—If we take equal lengths, say 20 or 30 feet, of wires of the same diameters but of different materials, as for example of copper and of German silver, wind first one and then the other around the frame of a tangent galvanometer and connect them successively to the plates of a galvanic cell, we shall find that only about $\frac{1}{15}$ as much current flows thru the German silver wire as flows thru the copper. This indicates that a given potential difference is capable of forcing far more current thru a copper conductor than thru an exactly similar one of German silver. German silver is therefore said to offer a higher *resistance* to the flow of electricity than does copper—its electrical *conductivity* is said to be lower. Silver has the highest conductivity of any substance known and therefore the lowest *resistivity*. The ratio of the resistance of a given piece of a wire of any material to the resistance of an exactly similar piece of silver wire is known as the *spe-*

cific resistance of the material. The specific resistances of a number of common conducting materials are given below:

Silver	1.00	Iron	6.00
Copper	1.11	Platinum	7.20
Aluminum	1.87	German Silver	15.20
	Mercury	63.1	

The specific resistance of a wire can be measured by measuring the current flow produced in it by a given difference of potential as described at the beginning of this paragraph. Similar experiments carried out with wires of any one material but of different lengths and cross areas would show the total resistance of any conductor to be directly proportional to its length and inversely proportional to the area of its cross section. The *unit* of resistance is the resistance of a column of mercury 106.3 centimeters long and 1 square millimeter in cross section at 0° C. This unit is called an *ohm* in honor of the German physicist Georg Simon Ohm (1787-1854). A piece of No. 30 copper wire (diameter .01 inch) 9.7 feet long has, at 20° C., a resistance of almost exactly one ohm. A more common size of copper wire, No. 16 (diameter .0508 inches) runs about 249.5 feet to one *ohm* resistance.

If, as is very convenient, we express the resistivity S of a material as the number of ohms resistance between opposite faces of a cube of the material, one centimeter long on each edge, then we can write for the total resistance R of any given conductor of length l and cross area a. $R = S\frac{l}{a}$. The resistance of a conductor depends on one other factor in addition to the factors involved in this formula, namely the temperature. The resistance of all metals increases with rise in tempera-

ture; the resistance of carbon and of liquid conductors decreases with rise in temperature.

10. Ohm's Law.—In 1826 Ohm first formulated the law showing the relation between the current delivered thru a given circuit, the resistance of the circuit and the value of the e. m. f. driving the current. *Ohm's Law* states that the current C furnished by different galvanic cells is directly proportional to the electro-motive force E existing in the circuit and inversely proportional to the resistance R of the circuit thru which the current flows. This may be written $C = \frac{E}{R}$—that is, the current flow expressed for instance in amperes in any circuit equals the e. m. f. expressed in volts in the circuit divided by the resistance expressed in ohms of the circuit itself. This is a simple law but has the widest and most important applications. In applying the formula it is important to notice that the R represents the total resistance in the closed path or circuit traversed by the current so that when this circuit includes a galvanic cell, the resistance of the liquid between the plates—called the " internal resistance " of the cell—must be added to the external resistances due to the connecting wires or other conductors before substituting for R. This fact is sometimes emphasized by writing the formula $C = \frac{E}{R\ (external) + r\ (internal)}$ or $C = \frac{E}{R + r.}$

Ohm's Law may also be applied to any part of a circuit as, for instance, to a single coil or wire in a circuit. The total difference of potential or drop in potential between the ends of the external circuit as between A and B, Fig. 102, equals the difference of potential between the plates of the cell. This total drop is made up of the drop from A to C plus that from C to D plus

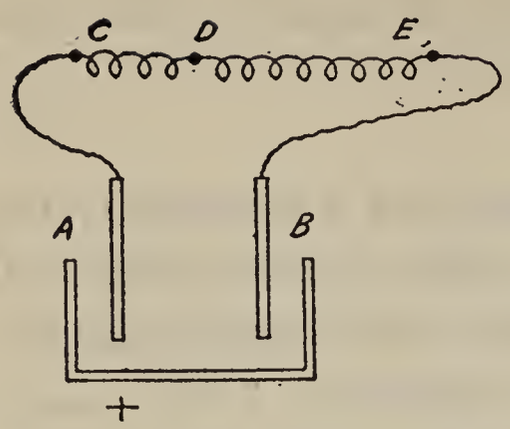

Fig. 102.

that from D to E plus that from E to B. The amount of potential difference (for short, *p. d.*) between C and D, for example, depends on the resistance between these points, being greater, the greater the resistance. This p. d. is also greater, the greater the current flow because of the larger electron densities involved in the larger current, all p. d.'s being primarily due to differences in electron concentrations. We have then $p.\ d. = R.\ C.$, which can be written $C = \frac{p.\ d.}{R.}$ or we can say, *the current flowing in the section of the circuit considered, measured in amperes, equals the potential difference between the ends of the section in volts divided by the resistance of the section in ohms,*—that is $C = \frac{p.\ d.}{R.}$ where *p. d.* represents the potential difference between the ends of the section. This is clearly a slightly modified form of Ohm's Law.

Ohm's Law may therefore be used to measure the resistance of a coil of wire. The coil marked X in Fig. 103 is connected with a galvanic cell B and an ammeter A in such a way (as shown in the figure) that the current flows thru the coil and then thru the ammeter, which indicates the strength of the current in amperes. The voltmeter V, connected across the ends of the coil indicates the drop in potential between the ends of the coil in volts. The voltmeter must have a very high resistance so that we may neglect the current which flows thru it because

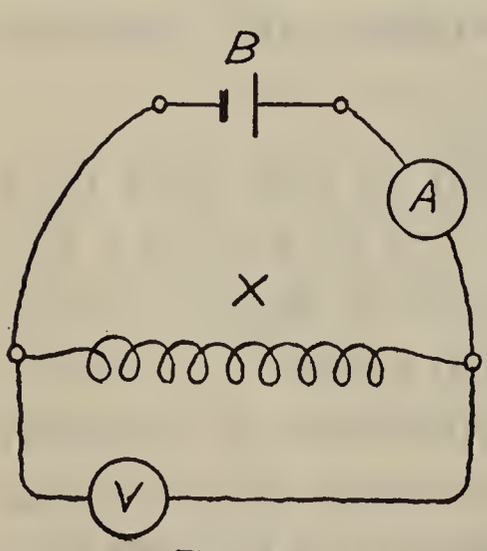

Fig. 103.

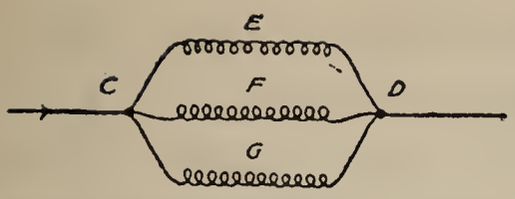

Fig. 104.

the ammeter shows the total current coming out of the battery, part of which goes thru X and part thru V. Under these conditions, neglecting the current flow thru V as being $\frac{1}{1000}$ or a less part of that thru X, if the ammeter shows 2.5 amperes and the voltmeter shows 4 volts, then the resistance of X is 1.6 ohms, since $R = \frac{p.\,d.}{C} = \frac{4}{2.5} = 1.6$. There are several other methods of measuring resistance which are in common use, but the one here described is the simplest.

11. Series and Parallel Connections.— If the conductors making up the external circuit are connected end to end as in Fig. 102, so that all the current delivered from the cell flows thru the conductors one after the other, the conductors are said to be connected in *series.* When so connected the total resistance from A to B clearly equals the sum of the individual resistances of the parts from A to C, C to D, D to E and E to B. Representing the total resistance by R_t we can write $R_t = R_1 + R_2 + R_3$ and so on. If the conductors are connected as in Fig 104, they are said to be connected in parallel. In this case, since the current has three paths to follow from C to D, the total resistance from C to D must be less than the resistance of any one of the conductors taken separately. If the three wires are exactly similar, the total resistance will equal $\frac{1}{3}$ that of one conductor, or $R_t = \frac{1}{3} R_1$, and if there are n conductors all alike, $R_t = \frac{1}{n} R_1$; since the conductivity of the three wires taken together is three times that of a single wire. If the wires are not all alike, the total conductivity C_t must always equal the sum of the individ-

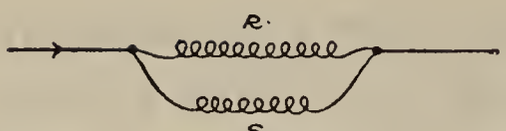

Fig. 105.

ual conductivities of the different branches; that is $C_t = C_1 + C_2 + C_3 + C_4$ and so on. Now, by definition, the conductivity of a wire is the reciprocal of its resistance, or $C = \frac{1}{R}$; whence we can write for resistances in parallel that $\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ and so on.

12. Shunts.—A wire connected like s in Fig. 105 around a resistance R so as to take part of the current which would otherwise flow thru R, is said to be a *shunt* to R. Under these conditions the current flowing from A will divide thru R and S in inverse proportion to the resistance of R and S. If the resistance of R is 1 ohm and that of S 9 ohms, then the current in R will be 9 times that in S, so that S takes $\frac{1}{10}$ of the total current and R takes $\frac{9}{10}$. Shunts are used around ammeter coils when currents are to be measured which are larger than those which the instrument is graduated to read.

REVIEW.

1. Describe a typical galvanic cell.
2. How are the phenomena of magnetism explained?
3. What is the cause of the declination of the compass needle? Of its dip?
4. How is a compass needle affected by being placed near a wire bearing a current of electricity?
5. Define conductivity and resistivity.
6. State Ohm's Law.

CHAPTER XV

ELECTRIC CURRENTS—THEIR EFFECTS

1. Heat Effects: Joule's Law.— We shall now turn to a somewhat detailed consideration of the effects of electric currents as applied to a great variety of industrial purposes. We shall consider, in turn, the *heating effects,* the *chemical effects* and the *magnetic effects* of currents.

When a current passes thru a wire, the electron motions produce an increase in the heat content of the wire and consequently its temperature rises. The amount of heat developed can be measured in calories by putting the wire into a calorimeter and applying the methods described in the chapter on Heat. It is found that the amount of heat developed is proportional to the *resistance* of the wire, to the *square* of the current strength and to the *time* the current flows. Putting W_H for the heat energy developed we have then $W_H \propto RC^2T$, or when the proper units are used, $W_H = RC^2T$. If the current is expressed in amperes, the resistance in ohms and the time in seconds, the heat energy will come out in joules. It will be remembered that a joule equals 10 million ergs of energy (that is 10 million dyne-centimeters) and that 4.2 joules are equivalent to one calorie of heat energy; consequently one joule is equivalent to .24 of a calorie.

The expression $W_H = RC^2T$ is known as *Joule's Law,* having first been announced by the English physicist James Prescott Joule (1818-1889). Since

by Ohm's Law $R = \frac{E}{C}$ Joule's Law may be written $W_H = \frac{E}{C} C^2T = ECT$, which shows that the number of joules of energy developed in a circuit in one second equals the product of the amperes flowing by the potential drop in volts thru the circuit. It will be remembered that a joule per second is a unit of *power* called a *watt,* so that in any circuit the *watts of power* expended equal the *product of the amperes in the circuit into the volts drop,* that is, watts = volts × amperes. The watt is therefore frequently referred to as the *volt-ampere;* and the kilowatt, which is the unit in which the power or time rate of doing work of large electrical generators is expressed, is often called the kilovoltampere or KVA.

In considering the energy of an electric current, it is useful to have in mind an analogy based on the flow of water thru a pipe. When a current has a strength of one ampere, it transfers a certain definite number of electrons, a certain definite quantity of electricity, past any point in the circuit in one second. The quantity carried past any point in each second, by a current of one ampere, is called a *coulomb.* This is the unit of *quantity* of electricity. If we represent quantity of electricity by Q, it is evident that $Q = CT$, wherefore, Joule's Law $W_H = ECT$ may be put into the third form, $W_H = EQ$. We can interpret this formula by saying that a coulomb of electricity, in passing from one point in a circuit to another point when a potential drop of one volt exists between the points, develops one joule of energy. This is analogous to saying that the energy developed by one pound of water moving in a pipe thru a drop of one foot is one foot pound. These formulæ: $W_H = RC^2T$, $W_H = ECT$ and $W_H = EQ$ enable us to calculate the energy developed in the form of heat in any circuit.

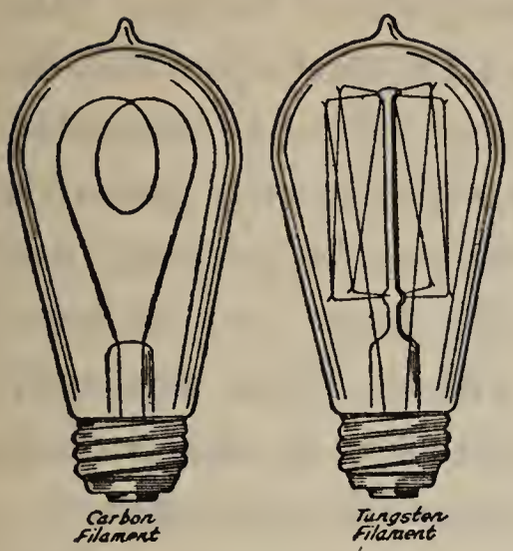

Fig. 106.

2. Electric Heaters.—The heating effect is made use of directly in street car heaters, in electric smoothing irons, bread toasters, stoves and similar devices. All such devices contain coils of high resistance wire which become very hot when a current is passed thru them. Special alloys which will not burn up or deteriorate from use have been developed for these coils. They are known by various trade names, as Nichrome, Advance, Reliance and so on, and generally are alloys of iron, nickel and chromium.

3. Filament and Arc Lamps.—The heating effect is made use of indirectly in various forms of electric lamps, which are of two general types—incandescent filament lamps and arc lamps. The filaments of incandescent lamps were formerly always made of carbon produced by carbonizing cotton. Such lamps, capable of giving 16 candle-power, take .5 ampere on a 110 volt circuit, using 55 watts, or about 3.4 watts for each candle-power. In recent years the filaments have been made almost exclusively of tungsten, which has the highest melting point of any metal known (3270° C.) and can be heated safely to a higher temperature than any other metal. The higher the temperature at which a filament is run the greater is the proportion of energy given out as light compared with the amount developed as heat—that is, developed in the form of radiation too long to affect the retina of the eye. Tungsten filament lamps are consequently much more efficient than carbon lamps because they can be run at a higher temperature.

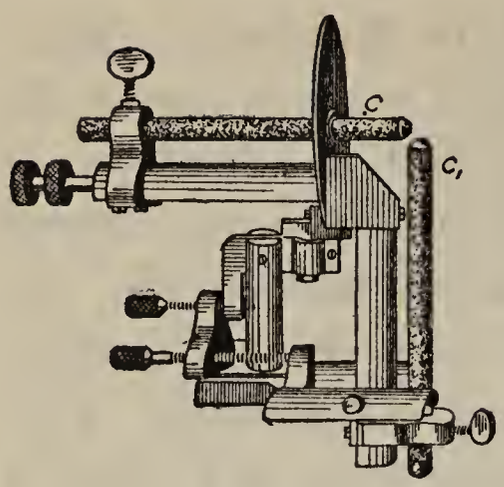

Fig. 107.

Mazda tungsten lamps take from 1 to 1.25 watts per candle. The filament of any incandescent lamp will burn up instantly if air or oxygen be present, and consequently they are inclosed in bulbs which either are completely evacuated or else contain some inert gas like nitrogen.

The ordinary arc lamp contains two carbon rods *C* and C_1, in Fig. 107, in the electrical circuit. To light the lamp these rods must be brought together and then drawn slightly apart. The heat developed at the loose contact vaporizes the carbon and a luminous discharge, which is known as the arc, is carried between the carbons in this vapor. The temperature developed at the end of the + carbon is estimated at 3800° C.—the highest temperature attainable by man. The arc lamp, having the highest temperature, is the most efficient form of electric lamp.

Lamps with solid carbon rods take about 10 amperes at 50 volts pressure and give about 1200 candle-power in the direction of maximum illumination. When the carbons are hollow and filled with a suitable mixture of lime, magnesia and similar substances, we have the so-called "flaming arc" with which efficiencies of .27 watt per candle-power are reached. Ordinary street arc lamps have automatic arrangements for drawing the carbons apart and maintaining them at the proper distance. These devices are usually somewhat unsteady in operation and cause the light to jump and flicker.

4. The Cooper-Hewitt Lamp.—The mercury vapor arc lamp, or Cooper-Hewitt lamp, consists of a long glass

tube with bulbs at either end containing mercury. The arc is started by tipping the lamp so that a thin stream of mercury runs from one end to the other. When this stream breaks, the mercury is vaporized and the vapor becomes incandescent and fills the entire tube. This lamp has a very high efficiency of .3 watt per candle. The light which it gives is entirely without red rays, so that objects seen under it generally appear to have unnatural colors. On account of the large proportion of blue and photographic rays emitted, this lamp has wide use in commercial photography.

5. Electrolysis.—The chemical effects of the electric current depend upon the fact that most chemically active substances when dissolved in a liquid *dissociate* into *ions,* each molecule, in itself as a whole uncharged, separating into one positively charged *ion* and one negatively charged *ion.* Thus the sulphuric acid molecule when dissolved in water forms + ions of hydrogen and — ions of SO_4; copper sulphate molecules give + ions of copper and — ions of SO_4, and so for other molecules. Ions of hydrogen and of the metals are + charged, other ions are — charged.

If plates, one positively and the other negatively charged by connection with a galvanic cell, are put into such a solution, the positive *ions* will be drawn to the negative plate and the negative *ions* to the positive plate, so that the constituents of the dissolved substance will be separated. This process of separation is called *electrolysis.* The charged plates put into the solution are called *electrodes,* the one positively charged being called the *anode* and the one negatively charged the *cathode.*

When plates of platinum—a substance in no way affected by sulphuric acid—are put into dilute sulphuric

acid and connected to a galvanic cell, hydrogen gas begins to collect on the negative plate at once. The SO_4 ions which travel to the positive plate give up their charges to it and then react chemically with the water, forming sulphuric acid again and liberating oxygen. This oxygen appears on the positive plate in the form of bubbles. Secondary chemical reactions, like this one, frequently go on at the electrodes of an electrolytic cell, so that the substances deposited on the plate are not necessarily the same as the ions contained in the liquid. If silver nitrate be dissolved in water, + silver ions and $-NO_3$ ions are formed. If now the charged platinum plates are introduced, silver will be deposited on the negative plate. This is the process of electroplating used in plating table ware and other articles—except that in practice some cheaper metal, not platinum, is used as the base for the silver deposit.

In 1834, Faraday found that a given current of electricity flowing for a given time deposits the same amount of a given element, no matter what may be the nature of the solution containing the element. One ampere flowing for one hour, for example, will in that period always deposit 4.025 grams of silver, 1.181 grams of copper and 1.203 grams of zinc from solutions of any compounds of these metals. The quantity of metal deposited in a given cell depending as it does on the current strength and on the time that the current flows, is accordingly governed by the quantity of electricity ($Q = CT$) that passes thru the cell. One coulomb will always deposit a certain definite amount of any given metal. This fact is used in the legal definition of the coulomb and the ampere. *The coulomb is the quantity of electricity required to deposit .001118 grams of silver; the ampere*

is the current which will deposit .001118 grams of silver in one second.

6. Polarization: Galvanic Cells.—The mode here described of transferring an electric current thru a liquid by the movement of oppositely directed streams of positive and of negative ions, is the one by which the current passes across from one plate to the other in a galvanic cell. When two plates are placed in a liquid and one dissolves faster than the other, the former becomes negatively charged. The negative plate of a galvanic cell is thus always the plate dissolved or " eaten " by the solution. In zinc—sulphuric acid—copper cells, the zinc plate is consequently the negative one. As soon as the plate is negatively charged, the + ions in the solution are drawn to it and the negative ions are driven to the other plate. The negative plate thus attracts the + hydrogen ions in the solution and becomes coated with a film of hydrogen gas. This cuts down the e. m. f. of the cell by substituting a plate of hydrogen for the zinc. The effect is known as *polarization.*

Different types of galvanic cells have been devised to reduce this effect as far as possible, usually by putting into the solution some substance, called a *depolarizer,* which combines with the hydrogen chemically. Manganese dioxide and potassium permanganate are often used for this purpose. The ordinary " doorbell battery " is a galvanic cell consisting of plates of zinc and of carbon in a solution of *sal-ammoniac* (ammonium chloride) which attacks the zinc but not the carbon. No depolarizer is used, as current is never taken continuously for any great length of time and the hydrogen gradually disappears from the zinc after standing. The ordinary " dry cell " contains a moist paste of sal-

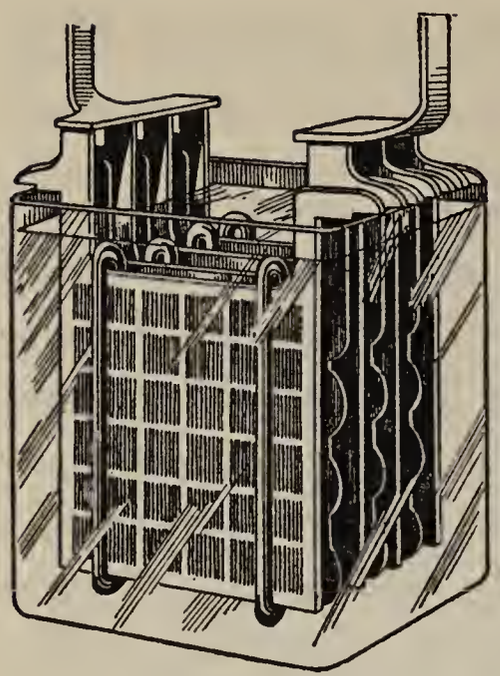

Fig. 108.

ammoniac and other materials between the outer zinc can and the inner carbon rod. When this paste dries up the cell ceases to develop an e. m. f. It is therefore not strictly a *dry* cell but only a *moist* cell. A depolarizer, manganese dioxide, is always used in these cells, since they are often required to deliver current more or less steadily.

7. The Storage Battery.—These chemical effects are used in electroplating, which we have already described, and in the storage battery. If plates of lead are put into dilute sulphuric acid, they are not affected by the acid. If these plates are charged + and — by connecting them to a number of galvanic cells, hydrogen will be released at the negative plate and oxygen at the positive. The hydrogen will not affect the negative plate, but the oxygen will convert the surface of the positive plate into lead oxide. If now the charging battery be removed, we have two dissimilar plates, one lead and the other lead oxide, in a chemically active solution; that is, we have a galvanic cell out of which current may be drawn. During the withdrawal of the current, the lead oxide breaks down and disappears entirely when the cell is discharged. The cell can then be recharged and used again. In practice the plates of storage cells are given certain special forms and are pasted over with certain mixtures of lead salts in order to increase their capacity for taking large charges. These cells when made of lead plates give an e. m. f. of about 2 volts. The plates are large and close together so that

very large currents may be drawn for short periods of time. Ordinarily such cells give back on discharge about 75 per cent of the energy put into them on charging.

8. **The Solenoid.**—We have already discussed several facts concerning the *magnetic effects* of currents. It will be remembered that a current-bearing conductor is surrounded by lines of magnetic force in the form of concentric circles about the wire. The direction of the lines, that is the direction in which a free north pole would move along them, is given by the Right Hand Rule. If the wire be bent into a coil, as in the tangent galvanometer, the concentric lines of magnetic force will pass into the plane of the coil from one side and pass out at the other. The effect, as far as the external field is concerned, is exactly as if the coil were a thin magnet in the form of a disc. If the wire be wound into a long cylindrical coil or helix—or " solenoid," as it is called,—by wrapping it around a wooden rod, the effect will be as if a number of magnetic discs were set face to face. The lines of force will enter one end of the solenoid and come out at the other, just as in a bar magnet. Such a solenoid, if suspended free to rotate, will act exactly like a compass needle.

9. **The Electromagnet.**—If a piece of soft iron be substituted for the wooden rod inside the solenoid, its molecules will be affected by the field of the solenoid and it will become an induced magnet. Such an arrangement of a soft iron bar inside a solenoid is called an *electromagnet.* As soft iron has a low retentivity, it ceases to be a magnet almost completely when the current in the solenoid is interrupted. Up to the limit of saturation, the strength of an electromagnet depends on

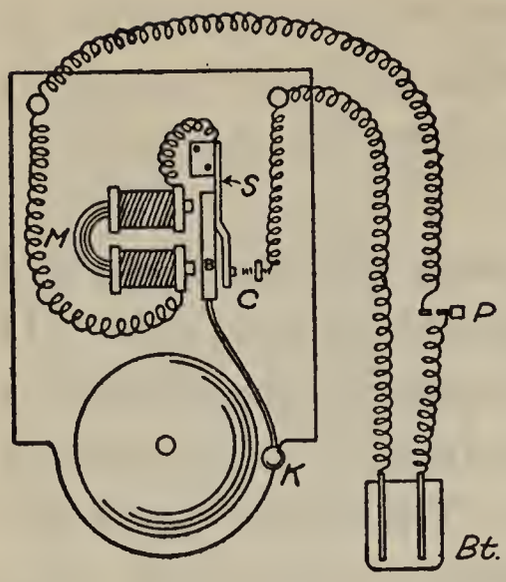

Fig. 109.

the strength of the inducing field in the solenoid, which clearly depends on the strength of the current in amperes and on the number of turns of wire per unit length. Two magnets which have equal numbers of "ampere turns" per centimeter will, speaking generally, have equal strengths—the "ampere turns" being the product of the number of turns of wire into the amperes flowing in the coil.

Electromagnets, when properly designed to carry heavy currents, can be made to lift very heavy loads of iron. Attached to cranes or derricks, they are widely used in foundries and similar places for picking up pig iron or castings. The most important use of the electromagnet, however, is in devices like the electric bell and the electric telegraph. The electric bell (Fig. 109) contains a horseshoe-shaped electromagnet with an iron bar or "armature" *B* supported across its poles by a spring *S*. When the button *P* is pushed down, the current from the cells *Bt* passes thru the coils of the magnet which, being thus rendered strongly magnetic, attracts the armature *B* and strikes the clapper *K* against the gong. When *B* moves over, the circuit is broken at *C*, which causes the magnet to lose its magnetism and *B* swings back under the influence of the spring *S*, closing the circuit at *C* again. The operation is then repeated, a rapid series of blows being struck on the rim of the gong as long as *P* is depressed.

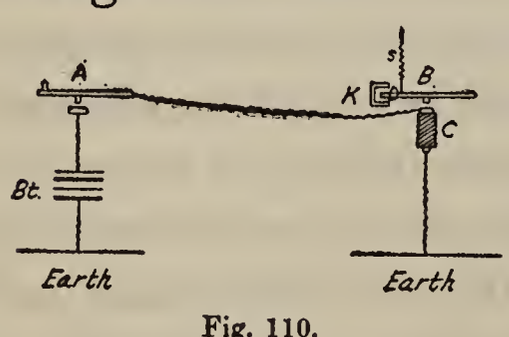

Fig. 110.

10. The Electric Telegraph.—The electromagnetic telegraph was invented about 1837 by Samuel F. B. Morse (1791-1872). It is, as shown in Fig. 110, an extremely simple application of the electromagnet. In order to save wire, a single line is used, the circuit being completed thru the earth. When the key at station *A* is depressed, the current from the battery *Bt* flows thru the coil *C* of the " sounder " at *B* and renders its core magnetic. The armature above it, which is pulled up against the upper stop of the clip *K* by the spring *S*, when no current is flowing, is drawn down against the lower stop when the current flows. It remains down as long as the key at *A* is closed.

Messages are sent in the well-known Morse code, which is built up of different combinations of " dots " and " dashes," representing the letters of the alphabet. A " dot " is sent when the armature remains down for a time, roughly half as long as that taken to send a " dash." An operator with a trained ear distinguishes dots and dashes by the interval between the click of the bar on the bottom stop and its return click on the upper stop. With telegraph lines many miles in length, the currents become so weakened by resistance that special sensitive " relays " must be used at the receiving station to detect the incoming signals. These are so lightly constructed that they do not make enough noise to be read by ear. Their armatures are therefore provided with extra contacts, which close local circuits containing local batteries, and " sounders " that make loud clicks. The first commercial telegraph line was between Baltimore and Washington and was opened by Morse on May 24, 1844. Many improvements have been made on the device here described, as a result of which several

messages can now be sent over the same wire in opposite directions at the same time.

11. Induced Currents.— We now come to a new and most important division of our subject—that of induced currents. If we thrust a bar magnet thru a coil of many turns of wire connected to a sensitive galvanometer, a momentary current is produced in the coil as evidenced by a deflection of the galvanometer needle. If the magnet be rapidly withdrawn, the needle will be deflected in the opposite direction. If, however, magnet and coil are kept stationary with reference to one another, no current flows thru the galvanometer. If the bar be held stationary and the coil be moved over it, currents will again be produced. Currents produced in this way are called *induced currents.* The experiments described can be extended to show that *an induced current is produced in a closed circuit whenever there is any change in the number of lines of magnetic force passing thru it,* however the change be produced, whether by movement of the coil relative to a magnet, by movement of a magnet relative to the coil, or by altering the magnetic field of an adjacent solenoid by varying the current flow in it. The discovery of these facts, made by the Englishman, Michael Faraday (1791-1867), in 1831, was one of the most far-reaching discoveries in the history of science. The most important modern developments of electricity are based on it.

The induced current produces its own magnetic field which, interacting with the moving inducing field, always tends to stop the motion of the inducing coil or magnet. This fact was first formulated by the physicist Lenz in 1834 and is known as Lenz's Law. The law is

Fig. 111.

merely a statement of the general principle of the conservation of energy in a new form; because if Lenz's law were not true, then we could develop electrical energy without expending an equivalent amount of mechanical energy—which is impossible according to the general principle.

We can always tell by the application of this law the direction of the induced current which will be produced by any given relative motion of a field and a conductor. The best simple rule for predicting the direction of the induced current is, however, *Fleming's Right-Hand Rule.* "Point the forefinger of the right hand in the direction of the lines of force and the thumb held at right angles to the forefinger in the direction of motion of the conductor—then the middle finger held at right angles to both forefinger and thumb will point in the direction of the induced current." (Fig. 111.) It is evident from these rules that if the direction of cutting the lines by the conductor be reversed, the direction of the induced current will also be reversed.

If the circuit in which the number of magnetic lines is changed is not a closed circuit, no induced current can flow; but a corresponding induced e. m. f. will be produced between the ends of the circuit. *The value of this induced e. m. f. has been found to depend only on the number of lines of magnetic force cut per second by the conductor.* Knowing this e. m. f., the value of the current induced, in case the circuit is closed, can be calcu-

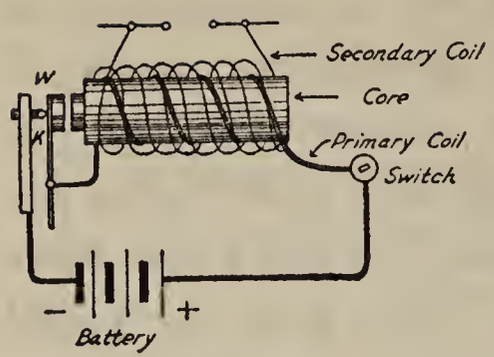

Fig. 112.

lated from Ohm's Law, when the resistance of the circuit is given. If, as is customary, we represent a magnetic field of unit strength —that is, a field in which at every point the force action on a unit pole is one dyne—by saying that it has one line per square centimeter, then a field of 100 units has 100 lines per square centimeter, and so on. If a conductor be moved thru a field in such a way that it cuts 100 million of these lines in each second, there will be induced between its ends an e. m. f. of one volt.

12. The Induction Coil.—One of the simplest devices producing induced currents is the induction coil. This consists of two coils of wire wound around the same core. The inner coil, called the primary coil, is of a few turns of coarse wire wound on an iron core; the outer or secondary coil consists of many turns of fine wire. The primary is connected (see Fig. 112) to a set of galvanic cells thru a "make-and-break piece" operated much like an electric bell by the magnetic attraction of the iron core. The vibrations of the weight *W* opening and closing the circuit at *K*, interrupt the primary circuit many times a second. When the circuit is closed, the current flows thru the primary and builds up a magnetic field around it; when the circuit is broken, this magnetic field collapses on the primary. While the field is building up, its lines pass outward thru the wires of the secondary coil and induce a current in it in one direction; while the field is collapsing, its lines cut the wires of the secondary oppositely and induce a current in the opposite direction. On account of the large number of turns—several thousands usually—in the secondary, the

value of the induced e. m. f. is very high, sufficient to produce long sparks between the secondary terminals. An interrupted low voltage direct or unidirectional current in the primary here produces a high voltage alternating current in the secondary—that is a high voltage current that flows first in one direction and then in the other. The number of watts of energy in the secondary circuit cannot be greater than, and in practice is never so great as, the number of watts in the primary circuit; that is $\text{Volts}_{\text{secondary}} \times \text{Amperes}_{\text{secondary}} \leq$ (equals or is less than) $\text{Volts}_{\text{primary}} \times \text{Amperes}_{\text{primary}}$. Since the voltage of the secondary current is so very much higher than that of the primary, its amperage must be correspondingly lower than that of the primary. The current output of the secondary of these coils is never very large.

13. The Dynamo.—We now know that when a wire is moved in a magnetic field so as to cut the lines of force in the field, an e. m. f. is induced between its ends. It is evident that if we could keep the wire moving across the field continuously, we would have a mechanical source of potential difference corresponding to a galvanic cell. This is what is done, as nearly as possible, in the *dynamo*. The dynamo consists of a large electromagnet *NS*, Fig. 113, called the "field magnet," in the strong field between the poles of which a rectangular coil called the "armature" is rotated on a shaft. The two ends of the coil are brought out to a pair of insulated rings R_1 and R_2 mounted on the shaft, as shown in the figure. With the coil rotating in the direction indicated by the arrowhead *a*, branch *A* is ascending thru the field and branch *B* is descending. The induced e. m. f.'s in these branches

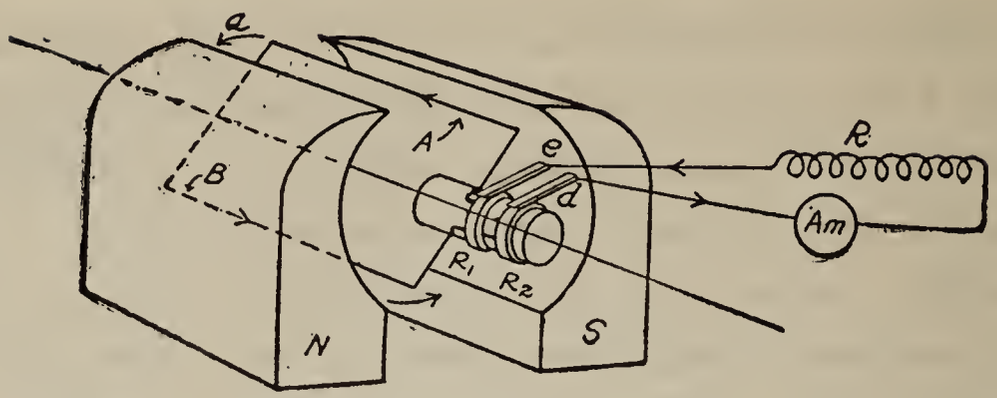

Fig. 113.

will be in the directions shown by the arrows on the coil, so that a current tends to flow out thru contact *d* and in thru contact *e*; that is to say, *d* is at a higher potential than *e*. The value of this e. m. f., depending as it does on the number of lines cut per unit of time, will be a maximum when the coil is cutting the field at right angles—that is, when it is horizontal in the figure—and will decrease to zero when the coil is vertical, because then for an instant the wire will be sliding along between the lines without cutting them.

After the coil passes the vertical position, *A* will begin to cut the field downward and *B* upward; the e. m. f. in the coil will therefore be reversed and *e* will be at a higher potential than *d*. This e. m. f. will increase to a maximum value when the coil is horizontal, and then decrease to zero—reverse in direction—increase to a maximum and so on. During one complete revolution of the coil from a position where it is vertical with *A* at the top until *A* takes the same position again after turning thru 360°, the potential of *d* passes thru the changes shown by the curve of Fig. 114, where positive potentials are drawn above the line AA^1 and negative potentials are drawn below. The current driven by this e. m. f. thru any external resistance, as *R*, Fig. 113, will alternate

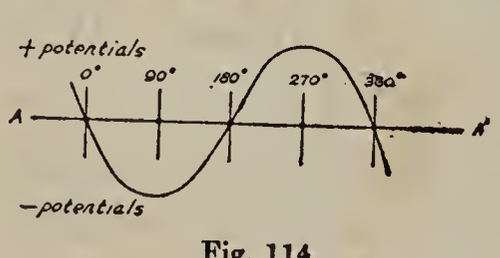

Fig. 114.

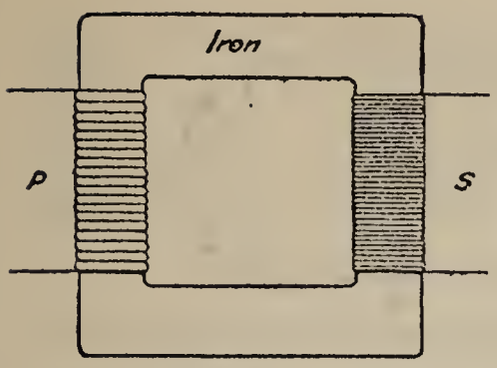

Fig. 115.

in direction, flowing first one way for one-half a revolution, and then the other way for the other half.

14. Alternating Current: the Transformer.—A dynamo built on this plan, delivering an alternating current, is called an alternator or alternating current generator. Currents of this type alternating 60 times a second are extensively used on electric lighting circuits. The voltages of such currents can easily be made higher or lower with little energy loss by the use of instruments known as transformers, Fig. 115. These consist of an iron core with two coils of wire wound on it; one a primary coil *P* in which the entering current flows, the other a secondary coil *S* out of which the delivered current is drawn. The action is the same as in an induction coil, excepting that no make-and-break piece is necessary. If the primary coil has fewer turns than the secondary, the output voltage will be higher than the input and the instrument is called a step-up transformer. If the secondary has fewer turns than the primary, the output voltage will be lower than the input and we have a step-down transformer.

In transmitting electrical power over long distances, it is economical to use very high voltages of 20,000 or 30,000 in order to cut down the loss due to heating in the line and to step it down with transformers to a suitable, usable voltage, ordinarily 110, near the place where it is to be employed. The best transformers have efficiencies as high as 98%—that is, they deliver 98 watts of energy out of each 100 watts put into them.

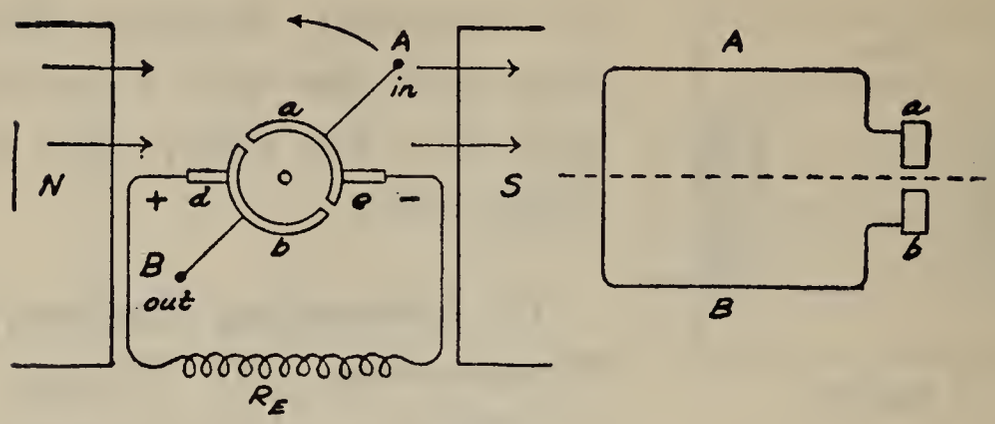

Fig. 116.

15. Direct Current.— There are some purposes for which alternating currents cannot be used. If unidirectional or direct current is wanted, the shaft of the dynamo is provided with a device known as a commutator, instead of the two collecting rings characteristic of an alternator. The two ends of the coil are rigidly connected to the two halves of a split ring, which rotates with the coil as shown in Fig. 116. A pair of fixed brushes fast to the frame of the machine slide on this split ring. With the rotation taking place in the direction shown, the current will be flowing toward the observer, that is "out," in the wire which is descending thru the field, and "in" in the wire ascending in the field. The brushes and ring are so arranged that brush *d* is always in contact with the descending wire and brush *e* with the ascending wire. The current consequently flows constantly out of *d*. The variation in the potential of *d* for one revolution of 360° is shown in Fig. 117. The curve representing the current in the external resistance will of course be of the same form. The current is unidirectional but it is not steady. To get steadiness it is necessary to use a large number of turns of wire instead of one and

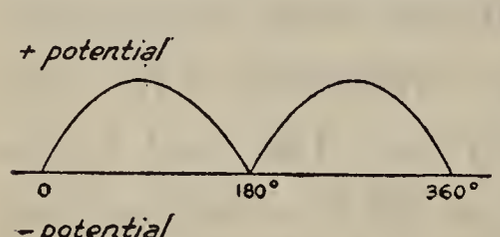

Fig. 117.

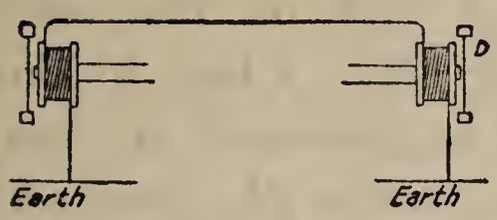

Fig. 118.

a commutator with a corresponding number of pairs of segments. A dynamo built with a commutator is called a direct current generator.

16. Direct Current Motor.—The direct current generator of Fig. 116 can be used as a *motor;* that is, if current is passed thru the coil from outside, the coil will rotate and will turn the shaft and do mechanical work. A conductor bearing a current in a magnetic field is acted on by a force due to the interaction of its own field with the other field. The direction of this force action is best remembered by what is known as the *Left-Hand Rule* or *Motor Rule.* "Point the forefinger of the left hand in the direction of the lines of force and the middle finger held at right angles to the forefinger, in the direction of the current sent thru the conductor, then the thumb held at right angles to both fingers will indicate the direction in which the conductor tends to move." The application of this rule to the coil in Fig. 116 will show that if current enters from an outside source thru brush *e, A* will be forced down and *B* up until the coil becomes vertical. At that instant the brushes slip across the gap in the commutator and the motion of the coil will be continued in the same direction, the brush *e* being always in contact with the wire descending thru the magnetic field. The motor here described is a *direct current motor.* Motors differing in construction from these are also made to operate on alternating current supply.

17. The Telephone.—The telephone is another important device using induced currents. The simple telephone (Fig. 118) consists of a permanent bar mag-

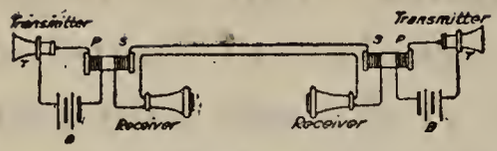

Fig. 119.

net with a coil of fine wire around one end. Close to this same end is mounted an iron diaphragm or disc *D*. One end of the coil is connected to the earth and the other to the line extending to the second station, at which there is an exactly similar instrument, similarly connected. When one speaks against the diaphragm, the vibrations in the air cause the diaphragm to vibrate. The vibrations of the diaphragm shift the magnetic lines of the field of the permanent magnet and induce corresponding e. m. f.'s in the coil wound round it. The resulting currents flowing in the coil of the distant telephone produce alterations in the strength of the magnet of that telephone, which cause the distant diaphragm to vibrate exactly or almost exactly as did the near diaphragm. Sound waves are thus produced by the distant diaphragm very similar to those which fall on the near diaphragm, and so speech is transmitted.

This type of telephone was invented in 1875 by Alexander Graham Bell and Elisha Gray. That simple device has been modified so as to work over long distances by introducing the microphone transmitter *T* in connection with a small transformer *SP*, as shown in Fig. 119. The microphone transmitter contains back of a diaphragm a loose contact with granules of carbon in it. The vibrations of the diaphragm cause variations in the resistance of the loose contact and thus vary the current from the battery *B*, passing thru the primary coil of the transformer *T*. The high voltage current developed in the secondary of the transformer is transmitted over the line and actuates the diaphragm of the receiver, which is still of the simple type described in the last paragraph.

REVIEW.

1. Define the coulomb.
2. Describe electrolysis.
3. How are induced currents produced?
4. Explain in general terms the principles of the dynamo.
5. Distinguish between direct and alternating currents.

INDEX

MATHEMATICS

A

Abscissa, 194.
Acceleration of gravity, 255–256.
Addition: algebraic, 129; of decimals, 5; of fractions, 23; of lines, 81; of mixed numbers, 23; of numbers, 4–5; of radicals, 149.
Algebra compared with arithmetic, 128–129.
Algebraic: addition, 129; division, 130–133; multiplication, 130; signs of trigonometric functions, 195; subtraction, 130.
Altitude, defined, 66.
Angle: acute, 53; between two arcs, 108; bisection of, 60; dihedral, 93; face, 97; horizontal danger, 59; inscribed, 59; obtuse, 53; plane, 93; polyhedral, 96; with the plane, 94; right, 53; trihedral, 96.
Angles: adjacent, 53; alternate interior, 56; complementary, 53; corresponding, of parallel lines, 56; equal, 58; measurement of, 29, 52, 191–192; negative, 191; opposite, 57; of spherical triangles, 109–110; supplementary adjacent, 53; symmetrical polyhedral, 97.
Antecedents in proportion, 77.
Antilogarithms, 180–181.
Apothem, 85.
Approximate computation, 11–14.
Approximation, 160.
Arcs: defined, 54; equal, 59; measurement of, 54.
Archimedes: 78, 86, 255; theorem of, 78.
Are, defined, 33.
Area: of circle, 85–86, 260; of cone, 103; between curves, 260–261; under a curve, 258–259; of frustrum of cone, 112; of parallelogram, 66–67; of polygon, 85; principle of, 66; of prism, 99; of pyramid, 101; of sphere, 113; of spherical triangle, 111–112; of surfaces of revolution, 262; of trapezium, 68; of trapezoid, 67–68; of triangle, 67.
Areas of similar triangles, 75–76.
Arithmetical: mean, 12; progression, 163–165.
Arithmetical series. *See* Arithmetical progression.
Austrian method of subtraction, 7.
Axis of pyramid, 100.
Axes of symmetry, 65.

B

Base: of a geometric figure, 66; of logarithms, 118; in percentage, 35, 36.
Binomial expansion, law of, 137–138.
Binomial theorem, 138, 170–171, 250.
Bisection: of an angle, 60; of an arc, 60; of a line, 60, 62–63.
Bisector of angle of triangle, 80.
Brokerage, 39.

C

Cancellation, 18–19.
Casting out nines, 19–20.
Centers, lines of, 62.
Characteristic of a logarithm, 178–179.
Chord, defined, 54.
Circle: arc of, 54; area of, 85–86; center of, 53; circumference of, 53; circumscribed, 65; definition of, 53; diameter of, 53–54; as envelope of set of chords, 90; inscribed, 65; as locus, 90–91; radius of, 53; sector of, 54; segment of, 54; tangent of, 54.
Circulating decimals, 166.
Circumference: of a circle, 53; degrees in, 54; radians in, 191.
Coefficient, 128.
Co-logarithms, 183.
Combinations, 168–169.
Commercial discount, 37–38.
Commission, 39.

Completing the square, 153.
Compound interest: 43–44; geometric series in, 167; by integration, 258; table, 45.
Computation: approximate, 11–14; logarithmic, 181–182; of logarithms, 252–254; primitive, 2.
Concurrent lines, 64.
Cone: lateral area of, 103; frustrum of—*see* Frustrum; volume of, 103.
Consequents, in proportion, 77.
Constant: of integration, 256; numbers, 120–121.
Continued proportion, 77.
Contracted methods of multiplication and division, 13–14.
Convergency, 172–174.
Coordinates, 226.
Cosecant, 195.
Cosine, 194.
Cost, 38.
Cotangent, 195.
Counting: primitive ideas of, 1–2; American system contrasted with British, 4.
Critical value, 247.
Cube: the, 98; volume of, 30.
Cube root, 147.
Curvature, 245.

D

Days: table of, for computing interest, 41.
Decimal point, use of, 4.
Decimals: addition of, 5; circulating, 166; division of, 11; logarithms of, 179; multiplication of, 8–9.
Degree, 192.
Denominator: defined, 21; least common, 23–24; rationalizing the, 148.
Derivative: the, 238; computation of, 240–241; as quotient of differentials, 245.
Diameter: of a circle, 53–54; of a sphere, 107–108.
Difference, 6.
Differential: the, 245; contrasted with derivative, 245; inverse of integral, 255; of independent variable, 245.
Differentiation: fundamental formulas of, 240–244; 249–251; successive, 245–246.
Dihedral angle, 93.
Dimensions, world of three, 48.
Discount, 37–38.
Divergency, 173–174.
Dividend, 9.
Division: algebraic, 130–133; of decimals, 11; of fractions, 25–27; of lines, 63, 80, 81, 82; contracted method of, 14; of mixed numbers, 26; of radicals, 150–151; rules of exact, 14–16; of numbers, 9–10; synthetic, 233–234.
Divisor: 9; common, 16.
Dodecahedron, 98.

E

Elimination, 221.
Ellipse, 226–227.
Envelope of set of chords, 90.
Equality of triangles, 55.
Equation: defined, 123; root of, defined, 162; use of, 125–128.
Equations: consistent, 220; of first degree, solution of, 218–223; of second degree—*see* Quadratic equations; higher, 158–159; dependent, 220; homogeneous, 160–161; inconsistent, 220; independent, 220; irrational, 161–162; in one unknown, 125–128, 152–164, 234–237; in more than one unknown, 217–233; relation of coefficients and roots, 237; systems of, 220–223; constant terms of, 220; theory of, 234–237. *See also* Simultaneous equations.
Equilateral: polygon, 68; rectangle, 58; triangle, 56, 64–65.
Equivalence: of pyramids, 102; of solids, 99–100; of spherical triangles, 110–111.
Eratosthenes' sieve, 15, 17.
Euclid: 50–51; Elements of, 51.
Exact division, rules of, 14–16.
Exact interest, 40, 43.
Exponents: defined, 118; difference between coefficients and, 128; laws of, 134–135.
External division of a line, 80–81.
Extremes in proportion, 77—*see also* Means.

F

Face angles, 97.
Factor, greatest common, 16–17, 143.
Factors: in multiplication, 7; prime, 16, 17, 18.
Factor theorem, 141–142.
Factoring, 138–141, 152–153.

Figures: non-rigid, 54–55; rigid, 54–55, 104; similar—*see* Similar. *See also* Equality, Equivalence.
Finite series, 163.
Formula: for arithmetical progression, 165; for cube root, 147; for geometric progression, 165–166; for lateral area of cone, 103; for lateral area of cylinder, 100; the prismoidal, 104; for right triangles, 72; for solving quadratic equations, 154–155; for square root, 144; three-side, 214–216; for volume of cone, 103; for volume of cylinder, 100; for volume of pyramid, 103.
Formulas: of differentiation, 240–244, 249–251; of integration, 263; fundamental trigonometric, 195–197; trigonometric, 207–215.
Four-step rule, 240.
Fractions: addition of, 23; defined, 21; division of, 25–27; improper, 21, 22, 23; in lowest terms, 22; multiplication of, 24–25; as new kind of numbers, 117–118; proper, 21, 22; subtraction of, 24; terms of, 21.
Frustrum: 101; of a cone, surface of, 112.
Functions: of a variable, 121–122; trigonometric—*see* Trigonometric.
Fundamental operations, 120.

G

Gain. *See* Profit.
Gauss, Carl Frederick, 85.
Geometric progression: 163–164, 165–166; in compound interest, 167.
Geometric series. *See* Geometric progression.
Geometric solid, defined, 47.
Geometry: compared with arithmetic, 47; origin of, 47; tools needed in the study of, 51–52.
Gerbert, Monk, 2.
"Golden Section," 82.
Graph: plotting a, 218–219; principle of, 219; use of, 219–220.
Graphical representation: 217; the circle, 225, 229, 232; the ellipse, 226–227; the hyperbola, 227–228; the parabola, 223; of adicals, 224–225; of roots, 235–236; of equations—*see* Graphical solution.
Graphical solution: of equations of first degree, 218–220; of quadratics (equations of second degree), 225–229, 232.
Great circle of a sphere, 107.
Greatest common factor, 16–17, 143.
Grouping similar terms, 138–139.

H

Harmonic: division of a line, 80; series, 164.
Hero of Alexandria, 214.
Hexahedron, 98.
Highest common factor, 16–17, 143.
Hindu-Arabic number system, 2–3.
Homologous: lines, 106; tetrahedrons, 105–106.
Hyperbola, 227–228.
Hypotenuse: defined, 57; the square on the, 70–71.

I

Icosahedron, 98.
Improper fractions: 21, 22; reducing, to mixed numbers, 23.
Infinite series, 163.
Integral contrasted with differential, 255.
Integration, 255; computing area by, 258–261, 262; computing interest by, 258; computing length by, 261; computing volume by, 261–262; fundamental formulas of, 263.
Interest: common, 40; compound, 43–44, 167, 258; table of compound, 45; exact, 40, 43; general method of computing, 42; rate of legal, 35; simple, 40; six per cent method of computing, 42–43; table of days for computing, 41.
Isosceles triangle, 57, 82–83.

K

Kite, the, 60–61, 62, 65, 76.

L

Lagrange, Joseph Louis, 162.
Least common denominator, 23–24.
Least common multiple, 17–18, 23, 143–144.

Length: 29; of curve by integration, 261; table of, 33. *See also* Measure, linear.
Limit of variable, 122, 124.
Line: bisection of, 60, 62–63; defined, 49; division of, into equal parts, 63; division of, into mean sections, 82; external division of, 80; internal division of, 80; harmonic division of, 80; oblique, projection of, 93–94; perpendicular to plane, 92–93; straight, 49, 90–91.
Lines: addition of, 81; division of, 81; multiplication of, 81; parallel—*see* Parallel; pencil of, 87–89; perpendicular, 53; perpendicular to plane, 94; distinction between planes and, 92; sets of, 87–89; subtraction of, 81; skew, 50.
Linkage, 59–60.
List price, 37.
Liter, 34.
Loci: intersection of, 89–90; of points, 87–89.
Locus, circle as, 90–91.
Logarithms: defined, 118; computation of, 252–254; of decimals, 179; invention of, 177; tables of, 184–185.
Long division, 10.
Longitude and time, 30.
Loss, 38.
Lune, 110.

M

Maclaurin's theorem, 249, 250–251.
Mantissa, 178.
Marked price. *See* List price.
Maxima and minima, 246–248.
Mean proportional, 77, 81–82.
Mean section, 82.
Means: arithmetical, 12, 164; in proportion, 77—*see also* Extremes.
Measure: angular, 29; circular, 29; cubic, 28; dry, 28; linear, 28; liquid, 28; of time, 29; square, 28. *See also* Measures, Metric System, Weight.
Measures, standard, 30, 33.
Measurement, impossibility of exact, 11–12.
Medians of a triangle, 64.
Merchiston, Baron, 177.
Metric system, 27, 32–34.
Minuend, 6.
Minus sign, explanation of, 117. *See also* Negative quality.
Mixed numbers: 21; addition of, 23; division of, 26; multiplication of, 25; reducing to improper fractions, 22–23; subtraction of, 24.
Monomial factor, removal of, 138.
Multiples, least common, 17–18, 23, 143–144.
Multiplicand, 7.
Multiplication: algebraic, 130; of decimals, 8–9; factors in, 7; of fractions, 24–25; of lines, 81; by logarithms, 181–182; of mixed numbers, 25; contracted method of, 13; of numbers, 7–8; of radicals, 149–150; table, 8.
Multiplier, 7.

N

Napier, John, 177.
Negative: angles, 191; direction, 191; quality, explanation of, 115–116—*see also* Minus sign; numbers, *see* Numbers.
Net price, 37.
Newton, Sir Isaac, 138, 255.
Nines, casting out, 19–20.
Non-rigid figures, 54–55.
Notation: primitive, 1–2; value of place in, 2.
Numbers: composite, 15; compound, 32; constant, 120–121; denominate, 27–32; even, 14; imaginary, 119; irrational, 118–119—*see also* Radicals; mixed, 21, 22–23, 24, 25, 26; negative, 114–115; odd, 15; positive integral, 114; prime, 15; transcendental, 120; unknown, 124; variable, 120–121—*see also* Variable, Variation; whole, 21.
Number system: Hindu-Arabic, 2–3; use of zero in, 3. *See also* Notation.
Numerator: defined, 21; rationalizing the, 148.

O

Oblique: line, projection on the plane, 93–94; prism, 99; triangle, 206–207, 211–212.
Octahedron, 98.
Ordinate, 193–194.

P

Pantograph, the, 75.
Parabola, 223.
Parallel: lines, 56, 59, 62–63, 78–79, 94, 95; planes, 50, 94–95; rulers, 58.
Parallelograms: 58; area of, 66; equivalent, 69.
Parallelopiped: 98; volume of, 99.
Pencil of lines, 87–89.
Pentagon, 57.
Percentage, 35.
Perimeter, 85.
Permutations, 167–168.
Perpendicular: the, 62; bisector, 60, 62, 64; lines, 94; to a plane, 92–93.
Pi (π): 86; value of, 251–252.
Place, 2, 5, 7, 11, 14, 129, 131–132, 146, 179, 249–254.
Plane: angle, 93; angle with the, 94; lines perpendicular to, 94; projection of oblique line on, 93–94; surface, 49.
Planes: distinction between lines and, 92; intersection of, 95; parallel, 50, 94–95.
Plus sign, explanation of, 117. *See also* Positive quality.
Points: defined, 49; of inflection, 246, 247; loci of, 87–89.
Polar triangle, 109.
Polygon: 55; area of, 85; perimeter of, 85.
Polygons: constructing, 84–85; regular, 84–85; similar, 83–84.
Polyhedral angles, 96.
Polyhedron, 98.
Positive quality, explanation of, 115–116. *See also* Plus sign.
Power: defined, 118; series, 249. *See* also Place.
Price, 37, 38.
Prime: factors, 16, 17, 18; numbers, 15; relative, 16.
Principal, 39.
Principle of permanence or no exception, 116–117, 120.
Prism: 98–100; area of, 99; volume of, 100.
Prismoids, 104.
Product: in multiplication, 7; special, 135–137.
Profit, 38.
Progressions, 164–166.
Proportion, 73, 77–79.
Proportional: line-segments, 95; mean, 77, 81–82.
Pyramids: defined, 100; equivalent, 102; triangular, 102–103, 105; truncated, 100–101.
Pythagorean theorem, 70–71, 76, 214, 224.

Q

Quadrant, 193.
Quadratic equations: defined, 152; formula for solution of, 154–155; graphical solution of, 225–229, 232; nature of roots of, 157–158; relation between roots of, 156–157; solution of, 152–156, 225–233.
Quadrilateral (=Quadrangle), 55, 56–57.
Quotient, defined, 9.

R

Radian, 191.
Radical expressions. *See* Radicals.
Radicals: 118–119, 147–151; graphical representation of, 224–225.
Radius: of a circle, 53; of a sphere, 107.
Radix, 27.
Range finders, principle of, 74.
Rate: of change, 245; in percentage, 35, 36–37.
Rates, 245.
Ratio: 73, 76–77; commensurable and incommensurable, 77.
Ratio of increments, 238–239.
Reciprocals, defined, 164.
Rectangular scale, use of, 63.
Rectangle, 58.
Regular polygons, 84–85.
Relative primes, 16.
Remainder, 9–10.
Removal of monomial factor, 138.
Repetend, 166.
Rhomboid, 58.
Rhombus, 58, 65.
Right: prism, 99; triangle, 202–206.
Rigid figures, 54–55, 104.
Rittenhouse, David, 14.
Root of an equation. *See* Equations.
Root: cube—*see* Cube; square—*see* Square.
Roots: defined, 118; complex, 157; equal, 157; graphical representation of, 235–236; irrational, 157; of quadratics, 156–158; rational, 157; real, 157; unequal, 157.

S

Secant, 54, 195.
Sector, 54.
Segment: 54; spherical, 113.
Selling price, 38.
Semicircle, 54.
Series: 163–166, 167, 172–175; power, 249.
Sets of lines, 87–89.
Short division, 10.
Significant figures, 12.
Signs: algebraic, of trigonometric functions—*see* Trigonometric; minus, 117; plus, 117; precedence of, 10.
Similar: figures, ratio of, 106; polygons, 83–84; solids, 104–106; terms, grouping, 138–139; triangles, 72–76, 79. *See also* Equality, Equivalence.
Simple interest, 40.
Simultaneous equations, 220–223, 228–234.
Sine: 194; law of, 211–212.
Six per cent method of computing interest, 42–43.
Slope: in graphs, 217; of tangent, 244.
Solids: defined, 49–50; of three dimensions, 50; similar, 104–106.
Solution: of equations—*see* Equations; of oblique triangle, 206–207, 211–212, 215–216; of right triangle, 203–206.
Space, properties of, 47–48.
Special products: 135–137; of radical expressions, 149–150.
Sphere: area of, 113; defined, 107; diameter of, 107–108; great circle of, 107; radius of, 107; small circle of, 107; tangent of, 107; volume of, 113.
Spherical: excess, 111–112; segment, 113; triangles, 108, 109–112; wedge, 110.
Square: defined, 58; equivalent to parallelogram, 71; equivalent to rectangle, 71.
Square root: 144–147, 153; of three, 171–172.
Stere, defined, 34.
Straight line: independent construction of, 90–91; defined, 49.
Subtraction: algebraic, 130; Austrian method of, 7; of fractions, 24; of lines, 81; of mixed numbers, 24; of numbers, 6–7; proof, 6.
Subtrahend, 6.
Sum, 4.
Surds. *See* Radicals.
Surface: 29; plane, 49. *See also* Area.
Swan pan, 2.
Sylvester II, Pope, 2.
Symmetry, 65.
Synthetic division, 233–234.

T

Tangent: 54; common external, 62; common internal, 62; how to draw a, 60–61; law of, 212–214; of a sphere, 107; trigonometric, 194.
Taylor's theorem, 248–250.
Terms: constant, of an equation, 220; grouping similar, 138–139.
Tetrahedron, 98, 104–105.
Theorem: binomial—*see* Binomial; the factor, 141–142; of Pythagoras—*see* Pythagorean.
Theory of equations, 234–237.
Time price—*See* List price.
Transversal, 78, 79.
Trapezium, area of, 68.
Trapezoid, area of, 67–68.
Triangular: prism, volume of, 100; pyramid, 102–103, 105.
Triangle: the, 54–55; area of, 67; concurrent lines in a, 64; exterior angle of, 56; equilateral, 56, 64–65; hypotenuse of, 57; interior angle of, 56; isosceles, 57, 82–83; oblique, 206–207, 211–212; polar, 109; right, 202–206; similar, 72–76, 79; spherical, 108, 109–112.
Trigonometric functions; 194–195; algebraic signs of, 195; of angles greater than 90°, 199–201; differentiation of, 249–251; of double angle and half angle, 209–210; inverse, 201; relations between, 195–197; of the sum of two angles, 207–209; sums and differences of, 210; tables of, 186–190; explanation of tables of, 182–183.
Trigonometric ratios, 183.
Trihedral angle, 96.
Truncated pyramid, 100–101.

U

Unknown quantities, defined, 124.

V

Variable: functions of, 121–122; infinite, 121; infinitesimal, 121; limit of, 122, 124; quantities, defined, 120–121.
Variation, direct, 217.
Velocity, 255–256.
Volume: defined, 29; of cone, 103; of cube, 30; of parallelopiped, 99; of prism, 100; of prismoid, 104; of revolution, integration of, 261–262; of sphere, 113; of triangular prism, 100; of triangular pyramid, 103.

W

Weight, avoirdupois, table of, 28.

Z

Zero: origin of, 3; use of, 3.
Zone, 113.

PHYSICS

A

Absolute: temperature, 387–388; zero, 376, 386, 387.
Adhesion, 328.
Air: brake, 314; compressibility of, 305; density of, 300; specific gravity of, 300.
Altimeters, 304.
Ammeter, 504.
Ampere, 504, 518–519.
Aneroid barometer, 303–304.
Angle: critical, 458; of incidence, 448; of reflection, 448.
Anode, 517.
Archimedes' Principle, 294–295, 297, 298, 299, 309, 322, 378.
Atmosphere: height of, 307–308; pressure of, 293, 300–303, 305–306. *See also* Weather predictions, Barometer.
Atom, 279, 492.
Attraction and repulsion, laws of, 490–491.

B

Balloon, 307, 308, 309–310.
Barograph, recording, 304.
Barometer: aneroid, 303–304; mercurial, 303; normal height of, 302–303; Torricelli's, 301–302. *See also* Atmosphere, pressure of; Weather predictions.
Beats, 433–435.
Bell, Alexander Graham, invention of, 532.
Boiling point, 401–402.
Boyle's Law, 306–307, 317, 386.

C

Caloric, 369.
Calorie, 391.
Calorimeter, 394–395.
Camera, 450–451, 473–474.
Candle power, 460–461.
Capillary action, 330–331.
Cathode, 517.
Centigrade scale, 372–374.
C. G. S. system (centimeter—gram—second), 282. *See also* Metric.
Charles' Law, 385–386, 387.
Chemical: elements, 279; rays, 487.
Chromatic aberration, 485–486.
Coefficient of expansion: of gases, 385–386; linear, 381–382; of liquids, 383–384.
Cohesion, 328.
Clouds, formation of, 322.
Cohesion, 332.
Cold storage plants, principle of, 404.
Colors: complementary, 480–481; mixed, 481–482; of opaque bodies, 479–480; and wave lengths, 478–479.
Cooper-Hewitt lamp, 480, 516–517.
Compass: 498; declination of, 501; dip of, 502.
Compensated: balance wheels, 382–383; pendulum, 383.
Compression in sound waves, 413–415.
Commutator, 530–531.
Condensation: 321–323; heat of, 400–401.
Conduction of heat, 376–377.
Conductivity, 507.
Conjugate foci, 470.
Connections: parallel, 511–512; in series, 511.
Conservation, doctrine of, 280–281, 360, 368, 393.

Convection, 377–379.
Coulomb, 514, 518.
Critical angle, 458.
Currents: alternating, 529; direct, 530–531; induced, 524–527.
Currents of the ocean, cause of, 378.
Cyclones, 305.

D

Density: defined, 279, 295; contrasted with specific gravity, 296; methods of determining, 295, 296–299; law of pressure and density of gases, 306–307; relation of, to weight and pressure in liquids, 291–292; table of, 297–298.
Depolarizer, 519.
Depth, relation of pressure to, 290–292.
Dew, formation of, 322.
Dewar's experiment, 375.
Dewpoint, 324.
Discords, cause of, 435.
Diving bell, 313.
Doctrine of conservation, 280–281, 360, 368, 393.
Doppler's Principle, 421–422.
Dynamics, defined, 286.
Dynamo, 527–531.
Dyne, 344, 353.

E

Echoes, 417–418.
Elasticity, 332–333.
Elements, chemical, 279.
Electric: bell, 522; heaters, 515; light bulbs, 515–516; telegraph, 523–524.
Electric currents: defined, 493, 496; magnetic effects of, 497, 502–504, 521–524; unit of, 504.
Electrical potential, 495–496.
Electricity: current, 489; frictional, explained by electron theory, 493–494; negative, 490, 492–493; positive, 490, 492–493; static, 489; unit of quantity of, 514.
Electrification: nature of, 489; kinds of, 490, 492–493.
Electrodes, 517.
Electrolysis, 517–519.
Electromagnet, 521–522.
Electrometer, 504–505.
E. M. F. (electro-motive force), 506–507.
Electrons: 279, 487, 492–493; distribution of, 494–495; theory of, 492–494.
Electroplating, 518.
Electroscope, 491–492.
Elements, chemical, 279.
Energy: 278, 280, 356–361: conservation of, 280–281, 360, 368, 393; kinetic, 357–360, 368; potential, 357–358.
English system of units, 282, 283.
Equilibrium: neutral, 351–352; stable, 351–352; unstable, 351–352.
Erg, 354.
Ether, 379–380, 451.
Evaporation: effect of heat on, 319–320; effect on temperature 323–324; explained by molecular motion, 319–321; in a vacuum, 325.
Expansion: of gases, 385; of liquids, 383–385; of solids, 381–383; coefficient of gas-, 385–386; coefficient of linear, 381–382; coefficient of liquid, 383–384.
Eye: defects of, 474–475; as optical instrument, 474–475.

F

Fahrenheit scale, 372–374.
Falling bodies, laws of, 348–351.
Faraday, Michael, discoveries of, 375, 518, 524.
Films, tension of surface, 326–328, 331.
Fleming's Right Hand Rule, 525.
Flotation, Law of, 298.
Focus, principal, 466.
Fog, formation of, 322.
Foot-pound, 354.
F. P. S. system (foot-pound-second) 282.
Force: 278; components of, 338–341; equilibrant of, 336–337; defined, 280; measurement of, 335; parallelogram of, 337, 338, 339, 340–341; resultant of, 327; 329, 330, 336, 337; units of, 283, 334–335, 344–345, 353.
Franklin, Benjamin, 495.
Fraunhofer lines, 483–484.
French system of units. *See* Metric system.
Friction, 357–358, 361–362, 366–367.
Fundamental note, 436, 439.

G

Galileo's: experiments with falling bodies, 349; telescope, 477; thermometer, 370.
Galvani, discovery of, 496.
Galvanic cells, 497, 519–520.
Galvanometers, 503–504.
Gas: engine, 407–409; use of illuminating, in balloons, 309; general law, 388–389.
Gases: compressibility of, 299, 385; expansion of, 385; mechanics of, 306–307.
Gay-Lussac's Law, 385–386, 387, 388.
Gilbert, Dr. William, discovery of, 489.
Gram: calorie, 391; force, 283, 334; mass, 283, 334.
Gravitation, Law of Universal, 347–348.
Gravity, center of, 351–352.
Gravity Pressure, Law of, 291, 299.
Gray, Elisha, invention of, 532.

H

Hail stones, formation of, 323.
Harmonics, 440.
Heat: absorption of, 395–396, 397, 403–404; of condensation, 400–401; conduction of, 376–377; energy in fuels, 405; of fusion, 397–398; induced by friction, 367; liberation of, 396, 397, 403–404; mechanical equivalent of, 391–392; molecular motion increased by, 319; nature of, 368; quantity of, 390; radiant, 380–381; specific, 392–394; transference of, 376–381.
Heat engines: 404–409; efficiency of, 408–409.
Heating systems, 379.
Helium, use of, in balloons, 310.
Helmholtz's resonators, 442.
Henry, Joseph, discovery of, 497.
Hertzian waves, 487.
Hooke's Law, 333, 335.
Horse-power, 355.
Humidity, 324–325.
Huygens, theory of, 451.
Hydraulic press, 289–290.
Hydrodynamics, defined, 286.
Hydrogen, use of, in balloons, 309–310.
Hydrometer, 298–299.
Hydrostatic balance, 297.
Hydrostatics, defined, 286, 287.

I

Ice, manufacture of, 404.
Illuminating power, 495.
Illumination, intensity of, 458–459,
Images: by concave lenses, 469, 472; by concave spherical mirror, 466–468; by convex lenses, 469–471; by convex spherical mirror, 465–466; by plane mirror, 463–465; by refraction, 468–472; real, 462, 473; virtual, 462, 473.
Incidence, angle of, 448.
Index of refraction, 455.
Induction coil, 526–527.
Inertia, 341–342.
Infra-red rays, 486.
Instruments: stringed, 438–440; wind, 437–438.
Insulators, 492.
Intervals, musical, 426–427.
Inverse Squares, Law of, 418–420, 459.
Ionization, 494–495.
Ions, 517.
Isobars, 304.

J

Joule, James Prescott, 367, 369, 513.
Joule's Law, 513–514.
Joule, a unit of work, 354.

K

Kilogrammeter, 354.
Kilovoltampere, 514.
Kilowatt, 356.

L

Lamp: arc, 515–516; carbon filament, 515; Cooper-Hewitt, 480, 516–517; tungsten filament, 515–516.
Law, defined, 284.
Left Hand Rule, 531.
Length: 282; units of, 282, 283.
Lens: achromatic, 486; chromatic aberration of, 485–486; concave, 468, 469, 472; convex, 468, 469–472; magnifying power of, 475; principal focus of, 469.
Lenz's Law, 524–525.
Lever, 362–363.
Loops of vibrating strings, 439.

Light: absorption of, 479–480; dispersion of, 478–479; intensity of, 458–459; interference of, 452–453; nature of, 444–445; rectilinear (straight line) propagation of, 450; reflection of, 447–448; refraction of, 454–457; speed of, 380, 445–446; wave length of, 454, 478–479; wave theory of, 451–452.
Lightning rods, principle of, 495.
Liquids: expansion of, 383–385; free surface of, 291, 329; mechanics of, 287–295. *See also* Pressure.
Luminous bodies, 444–445, 451.

M

Machines: efficiency of, 367; defined, 361; general law of, 361.
Magnetic: effects of electric currents, 497, 502–504, 521–524; field of force, 500–501; lines of force, 500, 502.
Magnetism: by induction, 499, 500; terrestrial, 501–502; theory of, 499–500.
Magnets: artificial, 498; natural, 498; saturated, 500.
Magnifying: glass, 475; power of a lens, 475.
Manometric flame, 441–442.
Mariotte's Law, 306–307.
Mass: 278–279, 282; units of, 282, 283, 334–335.
Matter: composition of, 279; conservation of, 280–281; defined, 278; gaseous, 286; liquid, 286; solid, 286.
Measurement: fundamental quantities of, 282; problems of, 281–282; systems of, 282–283. *See also* Units.
Mechanical advantage, 363, 364, 365, 366.
Mechanics, defined, 285.
Melting point, 398.
Mercury: barometer, 303; thermometer, 370–372, 374.
Method of mixtures, 394–395, 400–401.
Metric system, 282–283.
Microscope, compound, 475–476.
Mirror: concave spherical, 466–468; convex spherical, 465–466; plane, 463–465.
Mist, formation of, 322.
Molecular: forces in gases, 315–325; forces in liquids, 326–331; forces in solids, 331–333; motion, 316–317, 319, 368; theory, 315–333.
Molecules: 279; size of, 315; velocity of, 317, 319.
Momentum, 342–345, 346.
Morse, Samuel F. B., invention of, 523.
Morse Code, 523.
Motion: accelerated, 348–351; perpetual, 362; quantity of—*see* Momentum; uniform, 341–342.
Motor, direct current, 531.
Motor Rule. *See* Left Hand Rule.
Musical: chords, 427; instruments —*see* Instruments; intervals, 426–427; scale, 425–426, 427–429.

N

Newton's Law of Attraction, 347–348
Newton's Laws of Motion, 341–346.
Nodes of vibrating strings, 439.
Noise, 422–423.
Nucleus of the atom, 492.

O

Oersted's discovery, 497, 502–503.
Ohm's Law, 509–510, 526.
Ohm, a unit of resistance, 508.
Onnes, Kammerlingh, experiment of, 375, 386.
Opaque bodies, colors of, 479–480.
Opera glass, 477.
Optical instruments, 473–477.
Organ pipes, 430, 435–437.
Overtones, 436, 439–440, 441, 442.

P

Parallelogram of forces, 337, 338, 339, 340–341.
Pascal's experiment, 302.
Pascal's Law, 287–288, 299.
Penumbra, 449.
Perrier's experiment, 302–303.
Phonograph, principle of, 443.
Photographic rays, 487.
Photometer, 460–461.
Physics: defined, 278; value of, 277–278.
Pigments, mixed, 482.
Pitch, 429–430.
Plane: horizontal, 339–340; inclined, 340–341, 365–366.
Polarization, 519.

Potential difference: 504–506; unit of, 505.
Poundal, 344.
Pound force, 334–335.
Power: 355–356; units of, 355–356.
Pressure: of compressed air, 305; defined, 287; and density, law of—*see* Boyle's Law; and density, relation between, 306–307; effect of, on boiling point, 401–402; effect of, on freezing point, 399–400; external, 305–306; gravity, 290–291, 307; internal, 305–306; transmission of, in liquids, 287–288; relation between volume and, 306.
Prism binocular, 477.
Pulley, 364–365.
Pump: air, 311–312; force, 312–313; lift, 312.
Pyrometer, 374.

R

Radiation, 379–381.
Radiation spectrum, 486–487.
Rain, formation of, 322.
Rainbow, 484–485.
Rarefaction in sound waves, 414–415.
Reaction, 345–346.
Real images, 462, 473.
Reflection: angle of, 448; images formed by, 463–468; Law of Regular, 448; of light, 447–448; of sound, 416–418; total, 458.
Refraction: images formed by, 468–472; index of, 455; of light, 454–457.
Regular Reflection, Law of, 448.
Resistance, specific: 507–508; unit of, 508.
Resistivity, 507. *See also* Resistance specific.
Resonance, 430–433.
Right Hand Rule: 503; Fleming's, 525.
Roemer's experiment, 445–446.
Ross, Sir James, discovery of, 501.
Rumford, Count, experiments of, 369.

S

Saturation of vapor, 320–321.
Scales, musical, 425–426, 427–429.
Scientific method for determining facts, 284.
Screw, 366.
Shadows, 448–450.
Shunts, 512.
Siphon, 310–311.
Siren, 422–423, 425.
Snow, formation of, 322–323.
Solenoid, 521.
Solids, expansion of, 381–383.
Solutions, 403–404.
Sonometer, 438.
Sound: frequency of, 416; interference of, 430; intensity of, 418–420; musical, 422–424; origin of, 410–411; pitch of, 420–422; quality of, 441; reflection of, 416–418; transmission of, 411–413, 415–416; speed of, 415–416; vibrations of, 410–411; waves, 415–416.
Specific: gravity, 296, 297–298—*see also* Density; heat, 392–394; resistance, 507–508.
Spectroscope, 482–484.
Spectrum: radiation, 486–487; solar, 478, 483, 484.
Standard candle, 459.
State, changes of: liquids to gases, 400–403; solids to liquids, 395–400, 403.
Statics, defined, 285.
Steam engine: rotatory (steam turbine), 406–407; reciprocating, 405–406.
Storage battery, 520–521.
Surface: free, 291, 329; tension, 326–328, 331; units of, 283.

T

Telephone, principle of, 531–532.
Telescope, 475–476.
Temperature: absolute scale of, 387–388; absolute zero—*see* Absolute zero; Centigrade—*see* Centigrade; effect of evaporation on, 323–324; Fahrenheit—*see* Fahrenheit; meaning of, 369–370; effect of, on sound transmission, 415–416.
Tensile strength, 332.
Thales, 489.
Thermometer: alcohol, 371, 374; Centigrade, 372–374; Fahrenheit, 372–374; Galileo's, 370; hydrogen, 374–375; mercury, 370–372, 374; standard gas, 374–375.
Thermos bottle, principle of, 395.
Thermostats, 383.
Time: 282; units of, 282.
Torricelli, experiment of, 301–302.
Total reflection, 458.
Transformer, 529.

U

Ultra violet rays, 487.
Umbra, 449.
Units: of electric current, 504; of force, 283, 334–335, 344–345, 353; of heat quantity, 391; of length, 282, 283; of mass, 282, 283, 334–335; of measurement, 282; of momentum, 343; of potential difference, 505; of power, 355–356; of quantity of electricity, 514; of resistance, 508; of surface, 283; of time, 282; of velocity, 343; of volume, 283; of work, 354.

V

Vacuum: 301; evaporation in, 325; sounds in, 412; transmission of light in, 446.
Vapor: 320; saturation of, 320–321.
Vaporization: 400–401, 404; heat of, 400–401.
Variation: direct, 284–285; inverse, 284–285.
Velocity: 343, 348, 349–350; units of, 343.
Vibrating strings, laws of, 438–439.
Virtual images, 462, 473.
Volt, 505.
Volta, discovery of, 496–497.
Voltaic cell, 496–497.
Volt-ampere, 514.
Voltmeter, 507.
Volume: relation between pressure and, 306; units of, 283.

W

Water: behavior near freezing point, 384–385, 398–399; compressibility of, 289–290; maximum density of, 283, 384.
Watt, James, work of, 355, 405.
Watt, a unit of power, 355.
Wave theory of light, 451–452.
Weather predictions, 304–305. *See also* Atmosphere, pressure of; Barometer.
Wedge, 361.
Weight, 278, 279, 282.
Weston normal cell, 505.
Wheel and axle, 363–364.
Winds, cause of, 378.
Work: 353; units of, 354.

X

X-rays, 487.

Z

Zeppelins, 309.